

## PRELIMINARIES

### ✓ Sets

- A set is an unordered collection of elements. An element can be a number or another set. We denote sets with upper-case letters:  $A, B, C$ , etc.

- notation

$x \in A$  :  $x$  is an element of the set  $A$   
 $x$  belongs to  $A$ .

$x \notin A$  :  $x$  is not an element of the set  $A$   
 $x$  does not belong to  $A$ .

- Fundamentally, sets are defined via belonging condition of the form:

$$x \in A \Leftrightarrow p(x)$$

Here,  $p(x)$  is a predicate, i.e. a statement about  $x$  which is either true or false.

" $\Leftrightarrow$ " means that  $x \in A$  implies  $p(x)$  and  $p(x)$  implies  $x \in A$ .

### → Logic Operations

Let  $p, q$  be two statements that are either true or false. We define the following logic operations:

$p \wedge q$ :  $p$  true AND  $q$  true

$p \vee q$ :  $p$  true OR  $q$  true

(At least one of  $p$  or  $q$  is true, or both)

$p \nabla q$ : either  $p$  is true OR  $q$  is true, but not both.

$p \Rightarrow q$ : if  $p$  is true, then  $q$  is true

$p$  implies  $q$ .

$p \Leftrightarrow q$ :  $p$  is true if and only if  $q$  is true.

$p$  implies  $q$  and  $q$  implies  $p$ .

## → Quantified statements

Let  $A$  be a set and  $p(x)$  be a predicate. We define the following:

$\forall x \in A: p(x) \rightarrow$  For all  $x \in A$ ,  $p(x)$  is true

$\exists x \in A: p(x) \rightarrow$  There exists at least one  $x \in A$  such that  $p(x)$  is true.

## → Set algebra

- Sets can be defined by listing elements.

e.g.:  $A = \{2, 3, 6, 8\}$

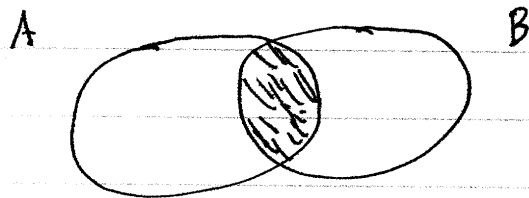
Note that elements cannot be repeated and the order of listing is not important. The corresponding belonging condition is:

$$x \in A \Leftrightarrow x=2 \vee x=3 \vee x=6 \vee x=8$$

- Let  $A, B$  be two sets. We define the following set operations; in terms of belonging conditions:

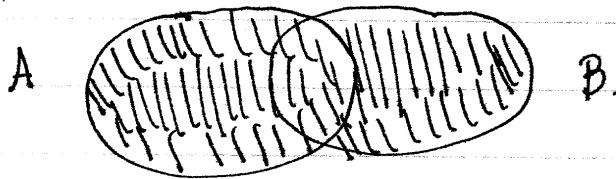
a) Intersection :  $A \cap B$

$$x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$$



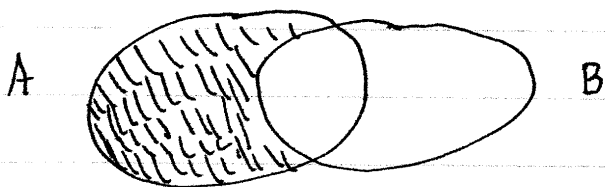
b) Union :  $A \cup B$

$$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B)$$



c) Difference :  $A - B$

$$x \in A - B \Leftrightarrow (x \in A \wedge x \notin B)$$



- Let  $A, B$  be two sets. We say that

a)  $A = B$  if and only if the sets  $A, B$  have the same elements.

b)  $A \subseteq B$  if and only if all the elements of the set  $A$  also belong to the set  $B$ .

These definitions more formally are written as follows:

$$A \subseteq B \Leftrightarrow \forall x \in A : x \in B$$

$$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$$

- We define  $\emptyset = \{\}$  as the empty set, i.e. a set with no elements.

### EXAMPLE

Let  $A = \{1, 2, 3, 4, 5, 6\}$  and  $B = \{2, 3, 7, 8\}$ . Evaluate the sets  $C = (A \cup B) - (A \cap B)$  and  $D = A - (A \cup B)$

Solution

$$\begin{aligned} C &= (A \cup B) - (A \cap B) = \\ &= (\{1, 2, 3, 4, 5, 6\} \cup \{2, 3, 7, 8\}) - (\{1, 2, 3, 4, 5, 6\} \cap \{2, 3, 7, 8\}) = \\ &= \{1, 2, 3, 4, 5, 6, 7, 8\} - \{2, 3\} = \\ &= \{1, 4, 5, 6, 7, 8\} \end{aligned}$$

$$\begin{aligned} D &= A - (A \cup B) \\ &= \{1, 2, 3, 4, 5, 6\} - (\{1, 2, 3, 4, 5, 6\} \cup \{2, 3, 7, 8\}) \\ &= \{1, 2, 3, 4, 5, 6\} - \{1, 2, 3, 4, 5, 6, 7, 8\} \\ &= \emptyset. \end{aligned}$$

## ▼ Number sets

- We define the following number sets:

1) Natural numbers :  $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

2) Integers:  $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$

3) Rational numbers:  $\mathbb{Q}$ .

A number  $x \in \mathbb{Q}$  if and only if there are two integers  $a \in \mathbb{Z}$  and  $b \in \mathbb{N} - \{0\}$  such that  $x = a/b$ .

i.e.:  $x \in \mathbb{Q} \Leftrightarrow \exists a \in \mathbb{Z} : \exists b \in \mathbb{N} - \{0\} : x = a/b$ .

4) Real numbers:  $\mathbb{R}$

The set  $\mathbb{R}$  of real numbers is informally defined as the set of all numbers that can be approximated by a convergent sequence of rational numbers.

- Let  $A$  be a set. and  $p(x)$  a predicate. We define

$$S = \{x \in A \mid p(x)\}$$

via the following belonging condition:

$$x \in S \Leftrightarrow (x \in A \wedge p(x)).$$

- We use this to define intervals:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\} \quad \left| \quad [a, +\infty) = \{x \in \mathbb{R} \mid a \leq x\}$$

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\} \quad \left| \quad (a, +\infty) = \{x \in \mathbb{R} \mid a < x\}$$

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\} \quad \left| \quad (-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$$

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\} \quad \left| \quad (-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$$

- In general, for a subset  $S \subseteq \mathbb{R}$ :

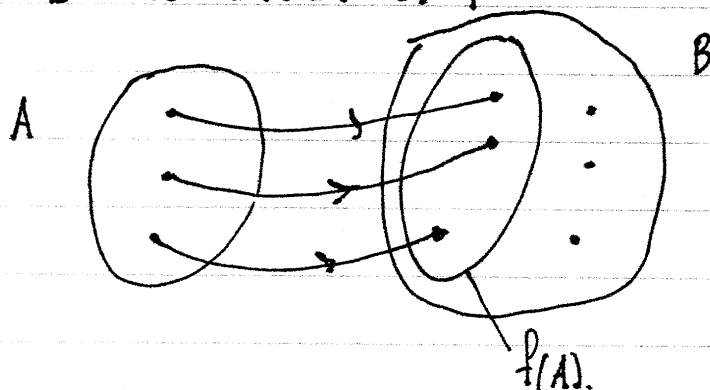
$$S \text{ interval} \Leftrightarrow \forall a, b \in S : (a < b \Rightarrow [a, b] \subseteq S)$$

## Mappings and Functions

- A mapping  $f: A \rightarrow B$  is a rule that maps every element  $x \in A$  to a unique element  $f(x) \in B$ .

$\text{dom}(f) = A \rightarrow$  domain of  $f$ .

$B =$  co-domain of  $f$



- A function  $f: A \rightarrow \mathbb{R}$  is a mapping with domain  $A \subseteq \mathbb{R}$  and co-domain  $\mathbb{R}$ .

### Range of a mapping / function

- Given a mapping  $f: A \rightarrow B$ , the range  $f(A)$  of  $f$  is the set of all elements of  $B$  for which there is at least one  $x \in A$  that maps to these elements. Formally, we define  $f(A)$  via the following belonging condition:

$$y \in f(A) \Leftrightarrow \exists x \in A : f(x) = y.$$

("  $y \in f(A)$  if and only if there is at least one  $x \in A$  such that  $f(x) = y$  ").

• notation : Given an expression  $\varphi(x)$  and a predicate  $p(x)$  and a set  $A$ , the set

$$S = \{\varphi(x) \mid x \in A \wedge p(x)\}$$

is defined via the following belonging condition:

$$y \in S \Leftrightarrow \exists x \in A : (p(x) \wedge y = \varphi(x))$$

Using this notation, we may rewrite the range definition as:

$$f(A) = \{f(x) \mid x \in A\}.$$

• In general, for  $S \subseteq A$ , we define

$$f(S) = \{f(x) \mid x \in S\}.$$

### EXAMPLE

Given the function  $f(x) = x^2 + 3$ ,  $\forall x \in \mathbb{R}$  and  $S = \{-1, 0, 1\}$  evaluate  $f(S)$ .

Solution

Since:

$$\left. \begin{array}{l} f(-1) = (-1)^2 + 3 = 1 + 3 = 4 \\ f(0) = 0^2 + 3 = 3 \\ f(1) = 1^2 + 3 = 1 + 3 = 4 \end{array} \right\} \Rightarrow$$

$$\Rightarrow f(S) = f(\{-1, 0, 1\}) = \{3, 4\}.$$

## → Defining a function / implied domain

- To define a function  $f$ , it is necessary to give both
  - a) The algorithm for  $f(x)$
  - b) The domain  $A$  of  $f$ .

This can be done succinctly using quantifier notation or verbally.

e.g: Let  $f$  be the function  $f(x) = x^3 + x^2 + 2x + 1$   
with domain  $A = \mathbb{R}$ .

OR more succinctly:

Define  $f(x) = x^3 + x^2 + 2x + 1, \forall x \in \mathbb{R}$ .

- If the domain is not given, then we assume the default domain to be the largest possible subset of  $\mathbb{R}$ .  
This is also called, the implied domain.

## EXAMPLES

- a) Find the implied domain of  
 $f(x) = (x^2 + 3x + 1)(x + 3)$

Solution

No restrictions, therefore  $A = \mathbb{R}$ .

↳ The default domain of any polynomial is always  $A = \mathbb{R}$ .

- b) Find the implied domain of

$$f(x) = \frac{3x+1}{x^3-4x}$$

Solution

Require  $x^3 - 4x \neq 0$ .

$$\text{Solve } x^3 - 4x = 0 \Leftrightarrow x(x^2 - 4) = 0 \Leftrightarrow x(x-2)(x+2) = 0$$

$$\Leftrightarrow x = 0 \vee x - 2 = 0 \vee x + 2 = 0 \Leftrightarrow x = 0 \vee x = 2 \vee x = -2$$

$$\Leftrightarrow x \in \{0, -2, 2\}.$$

It follows that  $A = \mathbb{R} - \{-2, 0, 2\}$ .

↳ For functions of the form  $f(x) = P(x)/Q(x)$  with  $P, Q$  polynomials, we require  $Q(x) \neq 0$ .

c) Find the implied domain of

$$f(x) = \sqrt{4-x} + \sqrt{x^2+3x+2}$$

Solution

Require: 
$$\begin{cases} 4-x \geq 0 \\ x^2+3x+2 \geq 0 \end{cases}$$

Solve:  $4-x \geq 0 \Leftrightarrow 4 \geq x \Leftrightarrow x \in (-\infty, 4] = S_1$

For  $x^2+3x+2 \geq 0$  we have:

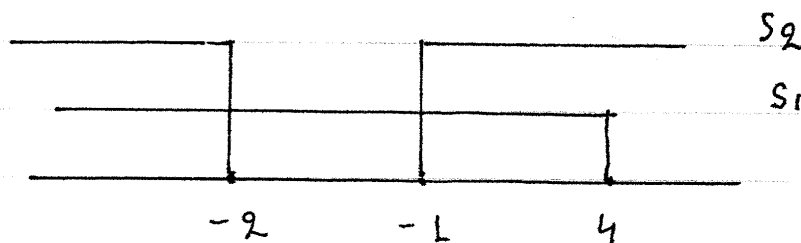
$$\Delta = b^2 - 4ac = 3^2 - 4 \cdot 1 \cdot 2 = 9 - 8 = 1 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-3 \pm \sqrt{1}}{2 \cdot 1} = \frac{-3 \pm 1}{2} = \begin{cases} -2 \\ -1 \end{cases}$$

|            |  |      |        |      |        |     |
|------------|--|------|--------|------|--------|-----|
| $x$        |  | $-2$ |        | $-1$ |        |     |
| $x^2+3x+2$ |  | $+$  | $\phi$ | $-$  | $\phi$ | $+$ |

It follows that

$$x^2+3x+2 \geq 0 \Leftrightarrow x \in (-\infty, -2] \cup [-1, +\infty) = S_2.$$



The domain is  $A = S_1 \cap S_2 = (-\infty, -2] \cup [-1, 4]$

↗ For functions containing terms of the form  $\sqrt{\varphi(x)}$  we require  $\varphi(x) \geq 0$ .

d) Find the implied domain of  $f(x) = \sin x + \tan(3x)$ .

Solution

Require that  $\forall k \in \mathbb{Z} : 3x \neq k\pi + \pi/2$

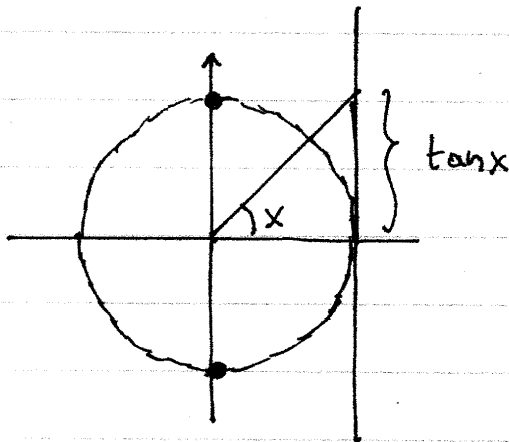
Solve  $3x = k\pi + \pi/2 \Leftrightarrow x = k\pi/3 + \pi/6$

It follows that  $A = \mathbb{R} - \left\{ \frac{k\pi}{3} + \frac{\pi}{6} \mid k \in \mathbb{Z} \right\}$

↗ For functions with terms of the form  $\tan(\varphi(x))$   
we require:  $\forall k \in \mathbb{Z} : \varphi(x) \neq k\pi + \pi/2$ .

Note that the domain of  $f(x) = \tan x$  is

$A = \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$ . On a trigonometric circle  
this means excluding the points at the top and  
bottom of the circle.



e) Find the implied domain of  $f(x) = \cot(\pi x + \pi/4)$ .

Solution

Require  $\forall k \in \mathbb{Z} : \pi x + \pi/4 \neq k\pi$ .

Solve:

$$\pi x + \pi/4 = k\pi \Leftrightarrow x + 1/4 = k \Leftrightarrow$$

$$\Leftrightarrow x = k - \frac{1}{4} = \frac{4k-1}{4}$$

It follows that  $A = \mathbb{R} - \left\{ \frac{4k-1}{4} \mid k \in \mathbb{Z} \right\}$

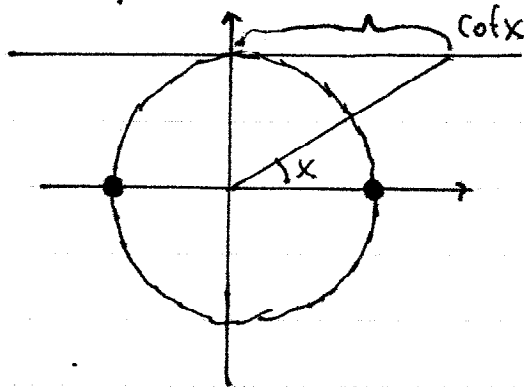
↪ For functions with terms of the form  $\cot(\varphi(x))$

we require:  $\forall k \in \mathbb{Z} : \varphi(x) \neq k\pi$ .

Note that the domain of  $f(x) = \cot x$  is

$$A = \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$$

On a trigonometric circle this means excluding the points on the left and right of the circle.



↪ See my College Algebra notes for review of solution methods on equations, inequalities, systems of inequalities, etc.

## EXERCISES

① Find the domain of the following functions.

$$a) f(x) = \frac{x}{x^2 - 6x + 9}$$

$$k) f(x) = \sqrt{\frac{x+2}{x-3}}$$

$$b) f(x) = \frac{x-1}{x^2 + 5x + 6}$$

$$l) f(x) = \frac{\sqrt{x+2}}{\sqrt{x-3}}$$

$$c) f(x) = \frac{(x+1)(x-2)}{(x-2)(x+2)}$$

$$m) f(x) = \frac{1}{\sqrt{9-x^2}} + \sqrt{x^2-4}$$

$$d) f(x) = \frac{x+1}{x+2}$$

$$n) f(x) = \sqrt{-x} + \frac{1}{\sqrt{5+x}}$$

$$e) f(x) = \sqrt{2x-5}$$

$$o) f(x) = \sqrt{(x^3-1)(x^2-5)}$$

$$f) f(x) = \sqrt{x^2 - x - 19}$$

$$p) f(x) = \sqrt{x^2-4} + \sqrt{x-1}$$

$$g) f(x) = \sqrt{x^2 + x + 3}$$

$$q) f(x) = \frac{2}{|x-2|-1}$$

$$h) f(x) = \sqrt{x^2 + 5x + 1}$$

$$r) f(x) = \sqrt{3-|x|}$$

$$i) f(x) = \sqrt{x-x^3}$$

$$s) f(x) = \sqrt{|x-2|-1}$$

$$j) f(x) = -\frac{2}{\sqrt{3-x}}$$

$$t) f(x) = \sqrt{|x|+3}$$

② Likewise, find the domain of the following functions.

a)  $f(x) = \tan(\pi x + x/3)$

b)  $f(x) = \tan\left(\frac{\pi(x+1)}{2} + \frac{3\pi}{2}\right)$

c)  $f(x) = \cot\left(\pi x + \frac{\pi(x+3)}{3}\right)$

d)  $f(x) = \cot\left(\frac{\pi}{6} + \frac{\pi x + 2\pi(x+2)}{5}\right)$