

IMPROPER INTEGRALS

Improper integrals of the first kind

Def : Let $f: A \rightarrow \mathbb{R}$ be a function such that
 $\forall b \in (a, +\infty)$: f integrable on $[a, b]$
Then, we define:

$$\int_a^{+\infty} f(x) dx = \lim_{b \rightarrow +\infty} \int_a^b f(x) dx$$

Def : Let $f: A \rightarrow \mathbb{R}$ be a function such that
 $\forall a \in (-\infty, b)$: f integrable on $[a, b]$
Then, we define

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

Def : Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function such that
 $\forall a, b \in \mathbb{R}$: $(a < b \Rightarrow f \text{ integrable on } [a, b])$
Then, we define:

$$\int_{-\infty}^{+\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{+\infty} f(x) dx$$

↗ In all of the above cases, we say that the corresponding improper integral converges if and only if the corresponding limit is equal to a real number. Otherwise, we say that the improper integral diverges.

↗ Notation

a) For $S = [a, +\infty)$:
$$\int_S f(x) dx = \int_a^{+\infty} f(x) dx$$

b) For $S = (-\infty, b]$:
$$\int_S f(x) dx = \int_{-\infty}^b f(x) dx$$

c) For $S = [a, b]$:
$$\int_S f(x) dx = \int_a^b f(x) dx$$

d) Likewise,
$$\int_{\mathbb{R}} f(x) dx = \int_{-\infty}^{+\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{+\infty} f(x) dx$$

► Simple improper integrals can be evaluated by calculating the corresponding finite integral and then taking the limit.

► The integral over $\mathbb{R}$ should be distinguished from the principle value integral over $\mathbb{R}$ which is defined as

$$\text{p.v.} \int_{\mathbb{R}} f(x) dx = \lim_{a \rightarrow +\infty} \int_{-a}^a f(x) dx$$

Note that the integral over $\mathbb{R}$ can diverge but the principle value integral over $\mathbb{R}$ can converge.

EXAMPLE

a) Evaluate the integral $I = \int_0^{+\infty} x e^{-x} dx$

Solution

We define:

$$\begin{aligned} f(x) &= \int_0^x t e^{-t} dt = \int_0^x (-t)(-e^{-t}) dt = \int_0^x (-t)(e^{-t})' dt = \\ &= [-t e^{-t}]_0^x - \int_0^x (-t)' e^{-t} dt = \\ &= -[x e^{-x} - 0 e^{-0}] + \int_0^x e^{-t} dt = -x e^{-x} + [-e^{-t}]_0^x = \\ &= -x e^{-x} - [e^{-x} - e^{-0}] = -x e^{-x} - [e^{-x} - 1] = 1 - e^{-x} - x e^{-x} \\ &= 1 - (x+1)e^{-x}, \quad \forall x \in (0, +\infty) \Rightarrow \\ \Rightarrow I &= \int_0^{+\infty} x e^{-x} dx = \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} [1 - (x+1)e^{-x}] = \\ &= 1 - \lim_{x \rightarrow +\infty} \frac{x+1}{e^x} = 1 - \lim_{x \rightarrow +\infty} \frac{(x+1)'}{(e^x)'} = \\ &= 1 - \lim_{x \rightarrow +\infty} \frac{1}{e^x} = 1 - 0 = 1. \end{aligned}$$

b) Evaluate the integral $I = \int_2^{+\infty} \frac{dx}{x \ln x}$

Solution

Define $f(t) = \int_2^t \frac{dx}{x \ln x}$, $\forall t \in (2, +\infty)$ and let

$$y = \ln x \Rightarrow dy = dx/x$$

It follows that

$$\begin{aligned} f(t) &= \int_{\ln 2}^{\ln t} \frac{dy}{y} = \left[\ln|y| \right]_{\ln 2}^{\ln t} = \ln|\ln t| - \ln|\ln 2| \\ &= \ln(\ln t) - \ln(\ln 2), \quad \forall t \in (2, +\infty) \end{aligned}$$

Since

$$\lim_{t \rightarrow +\infty} \ln t = +\infty \Rightarrow \lim_{t \rightarrow +\infty} \ln(\ln t) = +\infty \Rightarrow$$

$$\begin{aligned} \Rightarrow I &= \int_0^{+\infty} \frac{dx}{x \ln x} = \lim_{t \rightarrow +\infty} \int_0^t \frac{dx}{x \ln x} = \lim_{t \rightarrow +\infty} f(t) = \\ &= \lim_{t \rightarrow +\infty} [\ln(\ln t) - \ln(\ln 2)] = +\infty \Rightarrow \end{aligned}$$

$$\Rightarrow \int_0^{+\infty} \frac{dx}{x \ln x} \text{ diverges.}$$

c) Show that $\int_{\mathbb{R}} x^3 dx$ diverges, but $\oint_{\mathbb{R}} x^3 dx = 0$

Solution

Since,

$$\begin{aligned}\int_0^{+\infty} x^3 dx &= \lim_{a \rightarrow +\infty} \int_0^a x^3 dx = \lim_{a \rightarrow +\infty} \left[\frac{x^4}{4} \right]_0^a = \lim_{a \rightarrow +\infty} \frac{a^4}{4} \\ &= +\infty \Rightarrow \int_0^{+\infty} x^3 dx \text{ diverges} \Rightarrow \\ &\Rightarrow \int_{\mathbb{R}} x^3 dx \text{ diverges.}\end{aligned}$$

but

$$\begin{aligned}\oint_{\mathbb{R}} x^3 dx &= \lim_{a \rightarrow +\infty} \int_{-a}^a x^3 dx = \lim_{a \rightarrow +\infty} \left[\frac{x^4}{4} \right]_{-a}^a = \\ &= \lim_{a \rightarrow +\infty} \frac{a^4 - (-a)^4}{4} = \lim_{a \rightarrow +\infty} 0 = 0\end{aligned}$$

EXERCISES

① Evaluate the following integrals

$$a) I = \int_0^{+\infty} x \exp(-x^2) dx$$

$$b) I = \int_2^{+\infty} \frac{dx}{x^2+4}$$

$$c) I = \int_1^{+\infty} \frac{x dx}{(1+x^2)^2}$$

$$d) I = \int_0^{+\infty} x^2 e^{-x} dx$$

$$e) I = \int_0^{+\infty} e^{-x} \cos(2x) dx$$

$$f) I = \int_{-\infty}^{+\infty} \frac{dx}{x^2+2x+10}$$

$$g) I = \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2}$$

$$h) I = \int_1^{+\infty} \frac{dx}{x \sqrt{x^2+1}}$$

$$i) I = \int_2^{+\infty} \frac{dx}{x \sqrt{x^2-1}}$$

$$j) I = \int_2^{+\infty} \frac{dx}{x^2 \sqrt{x^2-1}}$$

$$k) I = \int_1^{+\infty} \frac{\ln x}{x^2} dx$$

$$l) I = \int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+9}} dx$$

Improper integrals of the second kind

Def : Let $f: A \rightarrow \mathbb{R}$ be a function such that

$$\forall t \in (a, b): f \text{ integrable on } [t, b]$$

Then we define

$$\int_{a^+}^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

Def : Let $f: A \rightarrow \mathbb{R}$ be a function such that

$$\forall t \in (a, b): f \text{ integrable on } [a, t]$$

Then, we define

$$\int_a^{b^-} f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

Def : Let $f: A \rightarrow \mathbb{R}$ be a function such that

$$\forall s, t \in (a, b): (s < t \Rightarrow f \text{ integrable on } [s, t])$$

and let $c \in (a, b)$ be given. Then, we define:

$$\int_{a^+}^{b^-} f(x) dx = \int_{a^+}^c f(x) dx + \int_c^{b^-} f(x) dx$$

→ It is easy to see that the last definition is independent of the choice of $c \in (a, b)$.

1 → We extend the definition of the notation

$$I = \int_S f(x) dx$$

where $S = (a, b]$ or $S = [a, b)$ or $S = (a, b)$ as follows:

$$\int_{(a, b]} f(x) dx = \int_{a^+}^b f(x) dx$$

$$\int_{[a, b)} f(x) dx = \int_a^{b^-} f(x) dx$$

$$\int_{(a, b)} f(x) dx = \int_{a^+}^{b^-} f(x) dx$$

EXAMPLES

$$a) I = \int_0^{1^-} \frac{dx}{\sqrt{1-x^2}}$$

Solution

Since

$$f(t) = \int_0^t \frac{dx}{\sqrt{1-x^2}} = [\operatorname{Arcsin} x]_0^t = \operatorname{Arcsin}(t) - \operatorname{Arcsin}(0) =$$

$$= \operatorname{Arcsin}(t) - 0 = \operatorname{Arcsin}(t) \Rightarrow$$

$$\Rightarrow I = \int_0^{1^-} \frac{dx}{\sqrt{1-x^2}} = \lim_{t \rightarrow 1^-} f(t) = \lim_{t \rightarrow 1^-} \operatorname{Arcsin}(t) =$$

$$= \operatorname{Arcsin}(1) = \pi/2$$

$$b) I = \int_{-(\pi/2)^+}^{(\pi/2)^-} \tan x \, dx$$

Solution

$$\text{Define } f(s, t) = \int_s^t \tan x \, dx, \quad \forall s, t \in (-\pi/2, \pi/2)$$

and let $y = \tan x$. It follows that

$$dy = (1 + \tan^2 x) dx = (1 + y^2) dx \Rightarrow dx = \frac{dy}{1 + y^2}$$

and therefore:

$$\begin{aligned}
 f(s,t) &= \int_{\tan(s)}^{\tan(t)} y \frac{dy}{1+y^2} = \frac{1}{2} \int_{\tan(s)}^{\tan(t)} \frac{2y}{1+y^2} dy = \\
 &= \frac{1}{2} \int_{\tan(s)}^{\tan(t)} \frac{(1+y^2)'}{1+y^2} dy = \frac{1}{2} \int_{\tan(s)}^{\tan(t)} [\ln(1+y^2)]' dy \\
 &= \frac{1}{2} \left[\ln(1+\tan^2 t) - \ln(1+\tan^2 s) \right] = \\
 &= \frac{1}{2} \ln \left(\frac{1+\tan^2 t}{1+\tan^2 s} \right)
 \end{aligned}$$

Let us consider

$$\int_0^{(\pi/2)^-} \tan x dx = \lim_{t \rightarrow (\pi/2)^-} f(0,t)$$

$$\begin{aligned}
 \text{Since } f(0,t) &= \frac{1}{2} \ln \left(\frac{1+\tan^2 t}{1+\tan^2 0} \right) = \frac{1}{2} \ln \left(\frac{1+\tan^2 t}{1+0} \right) = \\
 &= \frac{1}{2} \ln(1+\tan^2 t)
 \end{aligned}$$

and

$$\lim_{t \rightarrow (\pi/2)^-} \tan t = +\infty \Rightarrow \lim_{t \rightarrow (\pi/2)^-} \tan^2 t = +\infty \Rightarrow$$

$$\Rightarrow \lim_{t \rightarrow (\pi/2)^-} (1+\tan^2 t) = +\infty \Rightarrow$$

$$\Rightarrow \lim_{t \rightarrow (\pi/2)^-} \ln(1+\tan^2 t) = +\infty \Rightarrow$$

$$\Rightarrow \int_0^{(\pi/2)^-} \tan x dx = \lim_{t \rightarrow (\pi/2)^-} f(0,t) = +\infty \Rightarrow \int_0^{(\pi/2)^-} \tan x dx \text{ diverges}$$

$$\Rightarrow \int_{-(\pi/2)^+}^{(\pi/2)^-} \tan x dx \text{ diverges.}$$

EXERCISES

(2) Evaluate the following integrals or show they diverge

$$a) I = \int_0^{3^-} \frac{dx}{\sqrt{3-x}}$$

$$b) I = \int_{1^+}^2 \frac{dx}{(x-1)^{4/3}}$$

$$c) I = \int_0^2 \frac{x dx}{4-x^2}$$

$$d) I = \int_0^{1^-} \frac{3x dx}{\sqrt{1-x^2}}$$

$$e) I = \int_{-1^+}^0 \frac{dx}{x^2+3x+2}$$

$$f) I = \int_0^{1^-} \frac{dx}{x^3-x^2-x+1}$$

$$g) I = \int_0^{(\pi/6)^-} \tan(3x) dx$$

$$h) I = \int_{0^+}^{\pi/4} \frac{dx}{\sin x}$$

$$i) I = \int_0^{(\pi/2)^-} \frac{dx}{\cos^2 x}$$

$$j) I = \int_{\pi/4}^{(\pi/2)^-} \frac{\sin x}{1-\cos x} dx$$

$$k) I = \int_1^e \frac{dx}{x \ln x}$$

$$l) I = \int_1^e \frac{dx}{x(\ln x)^3}$$

$$m) I = \int_0^{(\pi/2)^-} \frac{dx}{1-\sin x}$$

$$n) I = \int_{0^+}^1 \frac{e^x}{\sqrt{e^x-1}} dx$$

Convergence criteria for improper integrals

- It is often possible to determine if an improper integral converges or diverges without evaluating it, by comparing it with other known integrals.
- To state the convergence criteria concisely, let $S \subseteq \mathbb{R}$ be the range of integration of an improper integral defined as a limit of the form

$$\int_S f(x) dx = \lim_{t \rightarrow \sigma} \int_{S(t)} f(x) dx$$

with $\sigma = a^+$ or $\sigma = a^-$ (with $a \in \mathbb{R}$) or $\sigma = +\infty$ or $\sigma = -\infty$ and with $S(t) \subseteq S$ where $S(t)$ is a closed interval. There are 4 possibilities:

S	$[a, +\infty)$	$(-\infty, b]$	$(a, b]$	$[a, b)$
$S(t)$	$[a, t]$	$[t, b]$	$[t, b]$	$[a, t]$
$t \rightarrow \sigma$	$+\infty$	$-\infty$	a^+	b^-

- Also recall that given any statement of the form $p \Rightarrow q$, proving this statement also proves the contrapositive statement $\bar{q} \Rightarrow \bar{p}$, where $\bar{p}, \bar{q}$ are the negations of p, q .

→ Convergence criteria

① Comparison test

$$\left. \begin{array}{l} \forall x \in S : 0 \leq f(x) \leq g(x) \\ \int_S g(x) dx \text{ converges} \end{array} \right\} \Rightarrow \int_S f(x) dx \text{ converges}$$

$$\left. \begin{array}{l} \forall x \in S : 0 \leq f(x) \leq g(x) \\ \int_S f(x) dx \text{ diverges} \end{array} \right\} \Rightarrow \int_S g(x) dx \text{ diverges}$$

→ Note that the second statement can be obtained as a contrapositive of the first statement.

② Ratio test

$$\left. \begin{array}{l} \forall x \in S : (f(x) \geq 0 \wedge g(x) \geq 0) \\ \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \in (0, +\infty) \end{array} \right\} \Rightarrow \left(\int_S f(x) dx \text{ converges} \Leftrightarrow \int_S g(x) dx \text{ converges} \right)$$

$$\left. \begin{array}{l} \forall x \in S : (f(x) \geq 0 \wedge g(x) \geq 0) \\ \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0 \end{array} \right\} \Rightarrow \left(\int_S g(x) dx \text{ converges} \Rightarrow \int_S f(x) dx \text{ converges} \right)$$

$$\left. \begin{array}{l} \forall x \in S : (f(x) \geq 0 \wedge g(x) \geq 0) \\ \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = +\infty \end{array} \right\} \Rightarrow \left(\int_S f(x) dx \text{ converges} \Rightarrow \int_S g(x) dx \text{ converges} \right)$$

↕ → The corresponding contrapositive statements are:

$$\left. \begin{array}{l} \forall x \in \delta: (f(x) \geq 0 \vee g(x) \geq 0) \\ \lim_{x \rightarrow \sigma} \frac{f(x)}{g(x)} \in (0, +\infty) \end{array} \right\} \Rightarrow \left(\int_{\delta} f(x) dx \text{ diverges} \Leftrightarrow \int_{\delta} g(x) dx \text{ diverges} \right)$$

$$\left. \begin{array}{l} \forall x \in \delta: (f(x) \geq 0 \wedge g(x) \geq 0) \\ \lim_{x \rightarrow \sigma} \frac{f(x)}{g(x)} = 0 \end{array} \right\} \Rightarrow \left(\int_{\delta} f(x) dx \text{ diverges} \Rightarrow \int_{\delta} g(x) dx \text{ diverges} \right)$$

$$\left. \begin{array}{l} \forall x \in \delta: (f(x) \geq 0 \wedge g(x) \geq 0) \\ \lim_{x \rightarrow \sigma} \frac{f(x)}{g(x)} = +\infty \end{array} \right\} \Rightarrow \left(\int_{\delta} g(x) dx \text{ diverges} \Rightarrow \int_{\delta} f(x) dx \text{ diverges} \right)$$

↕ → Note that the limit calculation can be interpreted as a comparison of growth rates of $f(x)$ and $g(x)$ as $x \rightarrow \sigma$.

Specifically we can give the following definitions:

a) $f(x)$ and $g(x)$ have equivalent growth rates as $x \rightarrow \sigma$:

$$f(x) \sim g(x) \text{ as } x \rightarrow \sigma \Leftrightarrow \lim_{x \rightarrow \sigma} \frac{f(x)}{g(x)} = 1$$

b) $f(x)$ has a slower rate of growth than $g(x)$ as $x \rightarrow \sigma$:

$$f(x) \ll g(x) \text{ as } x \rightarrow \sigma \Leftrightarrow \lim_{x \rightarrow \sigma} \frac{f(x)}{g(x)} = 0$$

The interpretation of the ratio test becomes obvious when you realize that convergence requires a sufficiently small growth rate.

③ Absolute convergence

$$\boxed{\int_S |f(x)| dx \text{ converges} \Rightarrow \int_S f(x) dx \text{ converges}}$$

→ The corresponding contrapositive statement is:

$$\boxed{\int_S f(x) dx \text{ diverges} \Rightarrow \int_S |f(x)| dx \text{ diverges}}$$

↪ Note that the absolute convergence criterion is often used in conjunction with the comparison test or with the ratio test.

→ Special Improper integrals

$$1) I = \int_a^{+\infty} \frac{dx}{x^p} \text{ converges} \Leftrightarrow p > 1 \quad (\text{with } a \in (0, +\infty))$$

$$2) I = \int_a^{+\infty} e^{-px} dx \text{ converges} \Leftrightarrow p > 0 \quad (\text{with } a \in \mathbb{R})$$

$$3) I = \int_{a^+}^b \frac{dx}{(x-a)^p} \text{ converges} \Leftrightarrow p < 1 \quad (\text{with } b \in (a, +\infty))$$

$$4) I = \int_a^{b^-} \frac{dx}{(b-x)^p} \text{ converges} \Leftrightarrow p < 1 \quad (\text{with } a \in (-\infty, b))$$

These results can be established by direct integral evaluation.

EXAMPLES

a) $I = \int_1^{+\infty} \frac{x dx}{3x^4 + 2x^2 + 3}$ $\leftarrow$ determine if I is convergent

Solution

1st method: Via Comparison test.

We claim that $\forall x \in (1, +\infty)$: $\frac{x}{3x^4 + 2x^2 + 3} \leq \frac{1}{3x^3}$ (1)

Let $x \in (1, +\infty)$ be given. Then:

$$x > 1 \Rightarrow x^2 > 1^2 \Rightarrow 2x^2 > 2 \Rightarrow 2x^2 + 3 > 2 + 3 = 5 \Rightarrow$$

$$\Rightarrow 3x^4 + 2x^2 + 3 > 5 + 3x^4 > 3x^4 > 0 \Rightarrow$$

$$\Rightarrow 3x^4 + 2x^2 + 3 > 3x^4 > 0 \Rightarrow$$

$$\Rightarrow 0 < \frac{1}{3x^4 + 2x^2 + 3} < \frac{1}{3x^4} \Rightarrow$$

$$\Rightarrow \frac{x}{3x^4 + 2x^2 + 3} < \frac{x}{3x^4} \quad [\text{because } x > 0]$$

$$\Rightarrow \frac{x}{3x^4 + 2x^2 + 3} < \frac{1}{3x^3}$$

and this proves the claim.

We also note that

$$\int_1^{+\infty} \frac{dx}{x^3} \text{ converges} \Rightarrow \int_1^{+\infty} \frac{dx}{3x^3} \text{ converges.} \quad (2)$$

From Eq. (1) and Eq. (2) via the comparison test, it follows that I converges.

2nd method : Via ratio test.

We define:

$$f(x) = \frac{x}{3x^4 + 2x^2 + 3}, \quad \forall x \in (1, \infty)$$

$$g(x) = \frac{1}{x^3}, \quad \forall x \in (1, \infty)$$

It follows that

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{\left(\frac{x}{3x^4 + 2x^2 + 3} \right)}{\left(\frac{1}{x^3} \right)} = \lim_{x \rightarrow \infty} \frac{x^4}{3x^4 + 2x^2 + 3} \\ &= \lim_{x \rightarrow \infty} \frac{x^4}{3x^4} = \frac{1}{3} \quad (1) \end{aligned}$$

We also note that $\int_1^{\infty} g(x) dx = \int_1^{\infty} \frac{dx}{x^3}$ converges (2)

From Eq. (1) and Eq. (2), via the ratio test, it follows that

$$I = \int_1^{\infty} f(x) dx \text{ converges.}$$

→ Note that the key insight in both methods is to recognize that the integrand has the same growth rate as $1/x^3$ in the limit $x \rightarrow \infty$. Also note that establishing the inequality, when using the comparison test, can be challenging, and requires a careful proof.

b) Show that $I = \int_0^{+\infty} e^{-x^2} dx$ converges.

Solution

1st method: Ratio test

We define:

$$f(x) = \exp(-x^2), \quad \forall x \in (1, +\infty)$$

$$g(x) = 1/x^2, \quad \forall x \in (1, +\infty)$$

and note that

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow +\infty} \frac{\exp(-x^2)}{1/x^2} = \lim_{x \rightarrow +\infty} x^2 \exp(-x^2) = \\ &= \lim_{x \rightarrow +\infty} \frac{x^2}{\exp(x^2)} = \lim_{x \rightarrow +\infty} \frac{(x^2)'}{[\exp(x^2)]'} = \lim_{x \rightarrow +\infty} \frac{2x}{(x^2)' \exp(x^2)} \\ &= \lim_{x \rightarrow +\infty} \frac{2x}{2x \cdot \exp(x^2)} = \lim_{x \rightarrow +\infty} \frac{1}{\exp(x^2)} = 0 \quad (1) \end{aligned}$$

$$\text{because } \lim_{x \rightarrow +\infty} x^2 = +\infty \Rightarrow \lim_{x \rightarrow +\infty} \exp(x^2) = +\infty \Rightarrow \lim_{x \rightarrow +\infty} \frac{1}{\exp(x^2)} = 0$$

$$\text{We also note that } \int_1^{+\infty} g(x) dx = \int_1^{+\infty} \frac{dx}{x^2} \text{ converges} \quad (2)$$

From Eq. (1) and Eq. (2), via the ratio test, it follows that

$$\int_1^{+\infty} e^{-x^2} dx \text{ converges} \Rightarrow \int_0^{+\infty} e^{-x^2} dx \text{ converges.}$$

↳ Note that for our convenience, we may restrict the range of integration in order to establish convergence.

→ This problem can be also done via the comparison test, using the following fundamental inequality from my Calculus I Lecture Notes:
 $\forall x \in \mathbb{R}: e^x \geq x+1$

2nd method : Comparison test.

Let $x \in [1, \infty)$ be given. Then:

$$e^{x^2} \geq x^2 + 1 > x^2 > 0 \Rightarrow \frac{1}{e^{x^2}} < \frac{1}{x^2} \Rightarrow e^{-x^2} < \frac{1}{x^2}$$

It follows that $e^{-x^2} < \frac{1}{x^2}$, $\forall x \in [1, \infty)$ (1)

We also note that

$$\int_1^{\infty} \frac{dx}{x^2} \text{ converges} \quad (2)$$

From Eq.(1) and Eq.(2), via the comparison test, we have:

$$\int_1^{\infty} \exp(-x^2) dx \text{ converges} \Rightarrow \int_0^{\infty} \exp(-x^2) dx \text{ converges.}$$

→ In general, the integral of $\exp(-x^2)$ cannot be evaluated in terms of elementary functions. It is in fact used to define a completely new function, the Gauss error function $\text{erf}(x)$ given by:

$$\forall x \in \mathbb{R}: \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$$

It can be shown that:

$$\lim_{x \rightarrow -\infty} \text{erf}(x) = -1 \quad \text{and} \quad \lim_{x \rightarrow \infty} \text{erf}(x) = 1.$$

c) Show that $I = \int_1^{+\infty} \frac{\ln x}{2x+3} dx$ diverges.

Solution

We define : $f(x) = \frac{\ln x}{2x+3}$, $\forall x \in (1, +\infty)$

$g(x) = \frac{1}{x}$, $\forall x \in (1, +\infty)$

and note that
 $\forall x \in (1, +\infty)$: $\frac{f(x)}{g(x)} = \frac{\left(\frac{\ln x}{2x+3} \right)}{\left(\frac{1}{x} \right)} = \frac{x \ln x}{2x+3}$ (1)

and

$$\lim_{x \rightarrow +\infty} \frac{x}{2x+3} = \lim_{x \rightarrow +\infty} \frac{x}{2x} = \frac{1}{2} \quad (2)$$

and $\lim_{x \rightarrow +\infty} \ln x = +\infty$ (3)

From Eq.(1) and Eq.(2) and Eq.(3): $\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = +\infty$ (4)

We also note that $\int_1^{+\infty} \frac{dx}{x}$ diverges. (5)

From Eq.(4) and Eq.(5), via the ratio test, it follows that

$$I = \int_1^{+\infty} \frac{\ln x}{2x+3} dx \text{ diverges.}$$

d) Show that $I = \int_{0^+}^1 \frac{dx}{x^2(x^2+3)}$ diverges.

Solution

→ Note that as $x \rightarrow 0^+$, in the denominator, the dominant term is the lowest-order term $3x^2$. Consequently, we must compare the integrand with $1/x^2$.

1st method : Ratio test

We define:

$$f(x) = \frac{1}{x^2(x^2+3)} = \frac{1}{x^4+3x^2}, \quad \forall x \in (0,1]$$

$$g(x) = \frac{1}{x^2}, \quad \forall x \in (0,1]$$

and therefore,

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x^2(x^2+3)}}{\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{x^2}{x^2(x^2+3)} = \\ &= \lim_{x \rightarrow 0^+} \frac{1}{x^2+3} = \frac{1}{0^2+3} = \frac{1}{3} \quad (1) \end{aligned}$$

From Eq.(1), via the ratio test, it follows that

$$\int_{0^+}^1 \frac{dx}{x^2} \text{ diverges} \Rightarrow I = \int_{0^+}^1 \frac{dx}{x^2(x^2+3)} \text{ diverges}$$

2nd method : Comparison test.

Let $x \in (0,1]$ be given. Since,

$$\begin{cases} 0 < x < 1 \Rightarrow x^2 > x^4 \\ 2 < 4 \end{cases}, \text{ and therefore,}$$

$$x^2(x^2+3) = x^4+3x^2 < x^2+3x^2 = 4x^2 \quad (1)$$

We also have:

$$x > 0 \Rightarrow \begin{cases} x^2 > 0 \\ x^2+3 > 0 \end{cases} \Rightarrow x^2(x^2+3) > 0 \quad (2)$$

From Eq.(1) and Eq.(2):

$$0 < x^2(x^2+3) < 4x^2 \Rightarrow \frac{1}{x^2(x^2+3)} > \frac{1}{4x^2}$$

$$\text{It follows that } \forall x \in (0,1]: \frac{1}{x^2(x^2+3)} > \frac{1}{4x^2} \quad (3)$$

From Eq.(3), via the comparison test, it follows that

$$\begin{aligned} \int_{0+}^1 \frac{dx}{x^2} \text{ diverges} &\Rightarrow \int_{0+}^1 \frac{dx}{4x^2} \text{ diverges} \\ &\Rightarrow \int_{0+}^1 \frac{dx}{x^2(x^2+3)} \text{ diverges} \end{aligned}$$

e) Show that $I = \int_1^{+\infty} \frac{\sin(\pi x)}{\pi x} dx$ converges.

→ It may seem that the integrand scales as $1/x$ and may therefore diverge. However, using integration by parts we can get growth rate $1/x^2$. Note that this problem also uses the absolute convergence theorem and the zero-bounded theorem.

Solution

We define

$$\begin{aligned} f(t) &= \int_1^t \frac{\sin(\pi x)}{\pi x} dx = \int_1^t \left(\frac{-\cos(\pi x)}{\pi} \right)' \frac{1}{\pi x} dx = \\ &= \left[\frac{-\cos(\pi x)}{\pi} \frac{1}{\pi x} \right]_1^t - \int_1^t \left(\frac{-\cos(\pi x)}{\pi} \right) \left(\frac{1}{\pi x} \right)' dx \\ &= - \left[\frac{\cos(\pi t)}{\pi^2 t} - \frac{\cos(\pi)}{\pi^2} \right] - \int_1^t \left(\frac{-\cos(\pi x)}{\pi} \right) \left(\frac{-1}{\pi x^2} \right) dx \\ &= \frac{-1}{\pi^2} - \frac{\cos(\pi t)}{\pi^2 t} - \int_1^t \frac{\cos(\pi x)}{(\pi x)^2} dx \quad (1) \end{aligned}$$

We need to show that all terms converge as $t \rightarrow +\infty$.

From the zero-bounded theorem (see my Calculus I online lecture notes)

$$\begin{cases} \forall t \in (1, +\infty): |\cos(\pi t)| \leq 1 \Rightarrow \lim_{t \rightarrow +\infty} \frac{\cos(\pi t)}{\pi^2 t} = 0 \\ \lim_{t \rightarrow +\infty} \frac{1}{\pi^2 t} = 0 \end{cases} \quad (2)$$

We also note that:

$$\forall x \in (1, +\infty): \left| \frac{\cos(\pi x)}{(\pi x)^2} \right| = \frac{|\cos(\pi x)|}{|(\pi x)^2|} = \frac{|\cos(\pi x)|}{(\pi x)^2} \leq \frac{1}{(\pi x)^2}$$

and it follows that

$$\int_1^{+\infty} \frac{dx}{x^2} \text{ converges} \Rightarrow \int_1^{+\infty} \frac{dx}{(nx)^2} \text{ converges} \Rightarrow$$

$$\Rightarrow \int_1^{+\infty} \left| \frac{\cos(nx)}{(nx)^2} \right| dx \text{ converges} \Rightarrow$$

$$\Rightarrow \int_1^{+\infty} \frac{\cos(nx)}{(nx)^2} dx \text{ converges.}$$

From Eq. (1) and Eq. (2):

$$I = \int_1^{+\infty} \frac{\sin(nx)}{nx} dx =$$

$$= \frac{-1}{n^2} - \lim_{t \rightarrow +\infty} \frac{\cos(nt)}{n^2 t} - \int_1^{+\infty} \frac{\cos(nx)}{(nx)^2} dx$$

$$= \frac{-1}{n^2} - \int_1^{+\infty} \frac{\cos(nx)}{(nx)^2} dx \Rightarrow \int_1^{+\infty} \frac{\sin(nx)}{nx} dx \text{ converges.}$$

EXERCISES

③ Without evaluating the following integrals, show whether they converge or diverge

a) $I = \int_0^{+\infty} \frac{3x dx}{x^5 + x + 1}$

b) $I = \int_0^{+\infty} x^p e^{-x^2} dx$, with $p > 0$.

c) $I = \int_0^{+\infty} \frac{x dx}{\sqrt{x^4 + 3}}$

d) $I = \int_2^{+\infty} \frac{dx}{x^2 \ln x}$

e) $I = \int_1^{+\infty} \frac{dx}{\sqrt{x} + \sqrt{x+1}}$

f) $I = \int_0^{+\infty} \frac{dx}{\sqrt{x^3 + 2}}$

g) $I = \int_e^{+\infty} \frac{dx}{\sqrt{x} \ln x}$

h) $I = \int_3^{+\infty} \frac{dx}{(x \ln x)^2}$

i) $I = \int_0^{+\infty} \frac{x dx}{e^x}$

④ Similarly, for the following integrals.

a) $I = \int_0^{1^-} \frac{x dx}{\sqrt{1-x^3}}$

b) $I = \int_{-1^+}^0 \frac{3x}{x^5 + 1} dx$

c) $I = \int_1^{4^-} \frac{x dx}{\sqrt{x} - 2}$

d) $I = \int_{0^+}^1 \frac{\ln x}{\sqrt{x}} dx$

e) $I = \int_{1/2}^{1^-} \frac{\ln x}{x-1} dx$

f) $I = \int_{0^+}^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

g) $I = \int_{0^+}^{1/2} \frac{\sqrt{x}}{\operatorname{Arcsin} x} dx$

h) $I = \int_{0^+}^1 \frac{dx}{x \operatorname{Arctan}(x)}$