

FIRST-ORDER ODEs

- A 1st-order ordinary differential equation (ODE) is an equation of the form $y' = f(x, y)$ satisfied by a function $y(x)$ of x . A corresponding 1st-order initial value problem is a problem of the form

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

with $x_0, y_0 \in \mathbb{R}$ given.

- An implicit solution to the initial value problem above is a solution of the form $F(x, y) = 0$ where we have shown that

$$\begin{cases} y' = f(x, y) \Leftrightarrow F(x, y) = 0 \\ y(x_0) = y_0 \end{cases}$$

- An explicit solution to the initial value problem above is a solution of the form $y = g(x)$ such that

$$\begin{cases} y' = f(x, y) \Leftrightarrow y = g(x) \\ y(x_0) = y_0 \end{cases}$$

- There is no general solution method that can give an implicit or explicit solution to a 1st-order ODE. However, solution methods exist for some special cases, including the following:

① → Separable ODEs

These are problems of the form

$$\begin{cases} y' = g(x)h(y) \\ y(x_0) = y_0 \end{cases} \quad (1)$$

Note that we say that

y_0 is a fixed point of Eq. (1) $\Leftrightarrow h(y_0) = 0$

If we initialize the system at a fixed point, then $y' = 0$, and we expect $y(x)$ to remain at the fixed point for all $x \in \mathbb{R}$. Furthermore, if we initialize at y_0 with $h(y_0) \neq 0$ then the solution cannot cross over any fixed point. We can therefore expect that $h(y(x)) \neq 0$ for all $x \in \mathbb{R}$ for which $y(x)$ can be obtained.

Methodology: Based on the above remarks we begin by assuming that $h(y) \neq 0$, and therefore:

$$\begin{aligned} y' = g(x)h(y) &\Leftrightarrow \frac{y'}{h(y)} = g(x) \Leftrightarrow \int \frac{dy}{h(y)} = \int g(x) dx \Leftrightarrow \\ &\Leftrightarrow H(y) = G(x) + C \end{aligned}$$

To determine C we use the initial condition $y(x_0) = y_0$:
 $H(y_0) = G(x_0) + C \Leftrightarrow C = H(y_0) - G(x_0)$.

Note that in the above argument we assume that the system has not been initialized at a fixed point. If the goal is to find a general solution, then it is necessary to explore whether the general solution continues to hold when y_0 is a fixed point.

EXAMPLES

a) Solve the initial value problem

$$\begin{cases} y'(x) = (1+y^2(x)) \cos x \\ y(0) = 1 \end{cases}$$

Solution

Since $1+y^2 > 0$, then the system has no fixed points.

We note that

$$y' = (1+y^2) \cos x \Leftrightarrow \frac{y'}{1+y^2} = \cos x \Leftrightarrow \int \frac{dy}{1+y^2} = \int \cos x \, dx \quad (1)$$

$$\text{with } \int \cos x \, dx = \sin x + C_1 \text{ and } \int \frac{dy}{1+y^2} = \text{Arctan}(y) + C_2$$

thus

$$(1) \Leftrightarrow \text{Arctan}(y) = \sin x + C \Leftrightarrow y = \tan(\sin x + C)$$

From the initial condition:

$$y(0) = 1 \Leftrightarrow \text{Arctan}(1) = \sin 0 + C \Leftrightarrow$$

$$\Leftrightarrow C = \text{Arctan}(1) = \pi/4$$

and therefore: $y(x) = \tan(\sin x + \pi/4)$.

We note that with increasing x , this solution becomes singular

when:

$$\sin x + \pi/4 = \pi/2 \Leftrightarrow \sin x = \pi/4 - \pi/2 \Leftrightarrow \sin x = \pi/4 \in [-1, 1]$$

$$\Leftrightarrow x = \text{Arcsin}(\pi/4).$$

► We say that the solution has a finite-time singularity at $x = \text{Arcsin}(\pi/4)$.

b) Solve the initial value problem

$$\begin{cases} y' = y^2 \\ y(0) = y_0 \end{cases}$$

Solution

We note that $y=0$ is a fixed point. We assume that initially $y_0 \neq 0$. Then $y \neq 0$, and it follows that

$$y' = y^2 \Leftrightarrow \frac{y'}{y^2} = 1 \Leftrightarrow \int \frac{dy}{y^2} = \int dx \Leftrightarrow \frac{y^{-1}}{-1} = x + C$$

$$\Leftrightarrow y^{-1} = -x - C \Leftrightarrow y = \frac{1}{-x - C} = \frac{-1}{x + C}$$

Since $y(0) = y_0 \Leftrightarrow y_0^{-1} = -0 - C \Leftrightarrow C = -y_0^{-1} = \frac{-1}{y_0}$
it follows that

$$y = \frac{-1}{x + C} = \frac{-1}{x - y_0^{-1}} = \frac{-y_0}{y_0(x - y_0^{-1})} = \frac{-y_0}{y_0 x - 1}, \text{ with } y_0 \neq 0$$

For the fixed point initialization $y_0 = 0$, the above equation correctly gives $y = \frac{-0}{0x - 1} = 0$, therefore it is valid

for all $y_0 \in \mathbb{R}$.

The solution has a finite time singularity when

$$y_0 x - 1 = 0 \Leftrightarrow y_0 x = 1 \Leftrightarrow x = 1/y_0.$$

c) Solve the initial value problem

$$\begin{cases} y' = 2x(y-1) \\ y(1) = y_0 \end{cases}$$

Solution

We note that $y-1=0 \Leftrightarrow y=1$, so $y=1$ is the fixed point. We assume initialization $y_0 \neq 1$, thus $y \neq 1$. Then,
 $y' = 2x(y-1) \Leftrightarrow \frac{y'}{y-1} = 2x \Leftrightarrow \int \frac{dy}{y-1} = \int 2x dx$

$$\Leftrightarrow \ln|y-1| = x^2 + C \quad (1)$$

From the initial condition

$$y(1) = y_0 \Leftrightarrow \ln|y_0-1| = 1^2 + C \Leftrightarrow C = \ln|y_0-1| - 1$$

and therefore

$$\ln|y-1| = x^2 + \ln|y_0-1| - 1 \Leftrightarrow$$

$$\Leftrightarrow |y-1| = \exp(x^2 + \ln|y_0-1| - 1) = \exp(x^2 - 1) \exp(\ln|y_0-1|) \\ = |y_0-1| \exp(x^2 - 1) \Leftrightarrow$$

$$\Leftrightarrow y-1 = \pm |y_0-1| \exp(x^2 - 1) \quad (2)$$

Since $y=1$ is a fixed point, for $y_0-1 > 0$ we will have $y-1 > 0$ and for $y_0-1 < 0$ we will have $y-1 < 0$. It follows that

$$(2) \Leftrightarrow y-1 = (y_0-1) \exp(x^2 - 1) \Leftrightarrow$$

$$\Leftrightarrow y = 1 + (y_0-1) \exp(x^2 - 1) \text{ for } y_0 \neq 1.$$

For $y_0=1$, the above solution gives $y=1$, so the general solution also works for $y_0=1$.

EXERCISES

(1) Solve the following initial value problems

$$a) \begin{cases} y' = x^3/y \\ y(1) = y_0 \end{cases}$$

$$b) \begin{cases} (1+x^2)y' = y \\ y(0) = y_0 \end{cases}$$

$$c) \begin{cases} y' + y^2 \cos x = 0 \\ y(0) = y_0 \end{cases}$$

$$d) \begin{cases} (y+1)y' = x^2 - 4 \\ y(0) = y_0 \end{cases}$$

$$e) \begin{cases} y' = \sqrt{1-y^2} \\ y(0) = y_0 \end{cases}$$

$$f) \begin{cases} e^{-x} y y' + x^2 = 0 \\ y(1) = y_0 \end{cases}$$

$$g) \begin{cases} y' = x y^3 (1+x^2)^{-1/2} \\ y(0) = y_0 \end{cases}$$

$$h) \begin{cases} y' = x^2 y \ln|x| \\ y(0) = y_0 \end{cases}$$

$$i) \begin{cases} y' = y^2 \arctan(x) \\ y(0) = y_0 \end{cases}$$

$$j) \begin{cases} \cos(2x) y' + \sin y = 0 \\ y(n/2) = n/3 \end{cases}$$

$$k) \begin{cases} y' = \sqrt{y^2 + 3y + 2} \\ y(0) = y_0 \end{cases}$$

$$l) \begin{cases} e^{-x} y' = y^{-1} \cos(2x) \\ y(1) = y_0 \end{cases}$$

$$m) \begin{cases} dy/dt = y^2 - 4 \\ y(0) = y_0 \end{cases}$$

→ For the solution of the above ODEs it may be necessary to review techniques of integration from Calculus 2.

② Logistic Population Model

The logistic population model is intended to model population growth under finite resources. If $y(t)$ is the population at time t , λ is the population growth rate, and N is the carrying capacity, then according to the logistic model, $y(t)$ is governed by

$$\frac{dy}{dt} = \lambda y (N - y)$$

Using initial condition $y(0) = y_0$, show that

$$y(t) = \frac{Ny_0}{y_0 + (N - y_0) \exp(-\lambda Nt)}$$

② → Homogeneous ODEs

Def: A homogeneous ODE is an equation of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Solution method: Let $y(x) = xu(x)$. It follows that:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \Leftrightarrow x \frac{du}{dx} + u = f(u) \Leftrightarrow x \frac{du}{dx} = f(u) - u \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{f(u) - u} \frac{du}{dx} = \frac{1}{x} \Leftrightarrow \int \frac{d\tilde{u}}{f(\tilde{u}) - \tilde{u}} = \int \frac{dx}{x} \Leftrightarrow \text{etc...}$$

EXAMPLE

Solve $\frac{dy}{dx} = \frac{2xy + y^2}{x^2}$ with $y_0 = -1/2$ for $x_0 = 1$.

Solution

We note that

$$\frac{dy}{dx} = \frac{2xy + y^2}{x^2} = \frac{2xy}{x^2} + \frac{y^2}{x^2} = 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \quad (1)$$

Let $y = xu \rightarrow u = y/x$. It follows that

$$(1) \Leftrightarrow x \frac{du}{dx} + u = 2u + u^2 \Leftrightarrow x \frac{du}{dx} = u^2 + 2u - u \Leftrightarrow$$

$$\Leftrightarrow x \frac{du}{dx} = u(u+1) \Leftrightarrow \frac{1}{u(u+1)} \frac{du}{dx} = \frac{1}{x} \Leftrightarrow$$

$$\Leftrightarrow \int \frac{du}{u(u+1)} = \int \frac{dx}{x} \quad (2)$$

Since $\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$ with

$$A = \frac{1}{u+1} \Big|_{u=0} = \frac{1}{0+1} = 1, \text{ and}$$

$$B = \frac{1}{u} \Big|_{u=-1} = \frac{1}{-1} = -1$$

it follows that

$$\begin{aligned} \int \frac{du}{u(u+1)} &= \int \left(\frac{1}{u} - \frac{1}{u+1} \right) du = \ln|u| - \ln|u+1| + C_1 \\ &= \ln \left| \frac{u}{u+1} \right| + C_1 \end{aligned}$$

and

$$\int \frac{dx}{x} = \ln|x| + C_2$$

and therefore

$$(2) \Leftrightarrow \ln \left| \frac{u}{u+1} \right| = \ln|x| + C \quad (3)$$

Apply the initial condition:

$$y(1) = -1/2 \Leftrightarrow u(1) = y(1)/1 = -1/2 \Leftrightarrow$$

$$\Leftrightarrow \ln \left| \frac{-1/2}{-1/2+1} \right| = \ln|1| + C \Leftrightarrow$$

$$\Leftrightarrow C = \ln \left| \frac{-1/2}{-1/2+1} \right| = \ln \left| \frac{-1}{-1+2} \right| = \ln|-1| = 0$$

and therefore:

$$(3) \Leftrightarrow \ln \left| \frac{u}{u+1} \right| = \ln|x| \Leftrightarrow \left| \frac{u}{u+1} \right| = |x| \Leftrightarrow$$

$$\Leftrightarrow \frac{u}{u+1} = x \vee \frac{u}{u+1} = -x \quad (4)$$

From the initial condition $u(1) = -1/2$ we note that $\frac{u}{u+1} < 0$ and $x > 0$, and therefore we reject

the first equation on (4) and have:

$$(4) \Leftrightarrow \frac{u}{u+1} = -x \Leftrightarrow u = -x(u+1) \Leftrightarrow u = -xu - x \Leftrightarrow$$

$$\Leftrightarrow (1+x)u = -x \Leftrightarrow u = \frac{-x}{1+x} \Leftrightarrow \frac{y}{x} = \frac{-x}{x+1}$$

$$\Leftrightarrow y = \frac{-x^2}{x+1}$$

EXERCISES

③ Solve the following homogeneous ODEs using initial condition $y(1) = y_0$.

a) $3xy' + y = x$

b) $(x-2y)y' = x+y$

c) $(x+3y)y' = 3x+y$

d) $x^2y' = y(x+y)$

e) $xy^2y' = y^3 - x^3$

f) $(x^2+y^2)y' = xy$

g) $xy' + y\sqrt{x^2 - y^2} = 0$

h) $y'\sqrt{x} = -\sqrt{x+y}$

④ Consider an ordinary differential equation of the form $M(x,y) + N(x,y)y' = 0$ such that

$$\forall \lambda \in (0, \infty): \begin{cases} M(\lambda x, \lambda y) = \lambda^a M(x, y) \\ N(\lambda x, \lambda y) = \lambda^a N(x, y) \end{cases}$$

with $a \in \mathbb{R}$.

a) Show that this ODE is homogeneous by reducing it to the form

$$\frac{dy}{dx} = \frac{-M(1, y/x)}{N(1, y/x)}$$

b) Show that the substitution $u = y/x$ reduces this ODE to the separable form:

$$\frac{1}{x} + \frac{N(1, u)}{M(1, u) + uN(1, u)} \frac{du}{dx} = 0$$

③ → Integrating Factors Method

This method can be applied to ODEs of the form:

$$y' + f(x)y = g(x)$$

with f, g continuous on $\mathbb{R}$.

Solution method

Define $h(x) = \exp\left(\int f(x) dx\right)$ and note that $h'(x) = f(x)h(x)$.

Then we multiply both sides of the ODE with $h(x)$:

$$y' + f(x)y = g(x) \Leftrightarrow y'h(x) + h(x)f(x)y = g(x)h(x) \Leftrightarrow$$

$$\Leftrightarrow y'h(x) + h'(x)y = g(x)h(x) \Leftrightarrow$$

$$\Leftrightarrow \frac{d}{dx}[yh(x)] = h(x)g(x) \Leftrightarrow$$

$$\Leftrightarrow h(x)y = \int h(x)g(x)dx + C$$

$$\Leftrightarrow y = \frac{1}{h(x)} \int h(x)g(x)dx + \frac{C}{h(x)} \quad (1)$$

→ Note that for $g(x) = 0$, the above solution simplifies to

$$y = \frac{C}{h(x)} = C \exp\left(-\int f(x) dx\right)$$

This is called the homogeneous term to Eq.(1).
The integral term is called the particular term.

EXAMPLE

a) Solve the ODE $y' + xy = x^2$ with $y(0) = y_0$.

Solution

Use the integrating factor

$$h(x) = \exp\left(\int x dx\right) = \exp(x^2/2) \Rightarrow h'(x) = xh(x)$$

and therefore:

$$\begin{aligned} y' + xy = x^2 &\Leftrightarrow y'h(x) + xh(x)y = x^2h(x) \Leftrightarrow y'h(x) + h'(x)y = x^2h(x) \Leftrightarrow \\ &\Leftrightarrow [yh(x)]' = x^2h(x) \Leftrightarrow yh(x) = c + \int_0^x t^2 h(t) dt \quad (1) \end{aligned}$$

$$\text{For } x=0: y_0 h(0) = c + 0 \Leftrightarrow c = y_0 h(0) = y_0 \exp(0) = y_0$$

and therefore,

$$(1) \Leftrightarrow yh(x) = y_0 + \int_0^x t^2 h(t) dt \Leftrightarrow$$

$$\Leftrightarrow y = \frac{y_0}{h(x)} + \frac{1}{h(x)} \int_0^x t^2 h(t) dt =$$

$$= \frac{y_0}{\exp(x^2/2)} + \frac{1}{\exp(x^2/2)} \int_0^x t^2 \exp(t^2/2) dt =$$

$$= y_0 \exp(-x^2/2) + \exp(-x^2/2) \int_0^x t^2 \exp(t^2/2) dt$$

→ The integrating factor method can be applied to the more general problem of the form

$$f(x)y' + g(x)y = h(x)$$

However, if $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then x_0 is a singular point of the ODE and the ODE will only yield a unique solution if x is restricted to an interval between neighboring singular points.

EXAMPLE

Solve the ODE $(x^2-1)y' + xy = 0$ with $y(x_0) = y_0$.

Solution

We have

$$(x^2-1)y' + xy = 0 \Leftrightarrow y' + \frac{x}{x^2-1}y = 0 \quad (1)$$

We will use the integrating factor

$$\begin{aligned} h(x) &= \exp\left(\int \frac{x}{x^2-1} dx\right) = \exp\left(\frac{1}{2} \int \frac{(x^2-1)'}{x^2-1} dx\right) = \\ &= \exp\left(\frac{1}{2} \ln|x^2-1|\right) = \exp(\ln\sqrt{|x^2-1|}) = \\ &= \sqrt{|x^2-1|} \end{aligned}$$

$$\Rightarrow h'(x) = h(x) \frac{x}{x^2-1} \quad . \quad \text{It follows that}$$

$$(1) \Leftrightarrow y' h(x) + \frac{x}{x^2-1} h(x) y = 0 \Leftrightarrow y' h(x) + y h'(x) = 0$$

$$\Leftrightarrow (d/dx) [y h(x)] = 0 \Leftrightarrow (d/dx) [y \sqrt{|x^2-1|}] = 0$$

$$\Leftrightarrow y \sqrt{|x^2-1|} = C \Leftrightarrow y = \frac{C}{\sqrt{|x^2-1|}}$$

We note that the ODE has singular points on $x=1$ and $x=-1$. From the initial condition:

$$y(x_0) = y_0 \Leftrightarrow \frac{C}{\sqrt{|x_0^2-1|}} = y_0 \Leftrightarrow C = y_0 \sqrt{|x_0^2-1|}$$

and therefore:

$$y = \frac{y_0 \sqrt{|x_0^2-1|}}{\sqrt{|x^2-1|}}$$

We distinguish between the following cases:

Case 1 : If $x_0 \in (-\infty, -1)$, then $|x_0^2-1| = x_0^2-1$ and

$$y = \frac{y_0 \sqrt{x_0^2-1}}{\sqrt{x^2-1}}, \quad \forall x \in (-\infty, -1)$$

Case 2 : If $x_0 \in (-1, 1)$, then $|x_0^2-1| = 1-x_0^2$ and

$$y = \frac{y_0 \sqrt{1-x_0^2}}{\sqrt{1-x^2}}, \quad \forall x \in (-1, 1)$$

Case 3 : If $x_0 \in (1, \infty)$, then $|x_0^2-1| = x_0^2-1$ and

$$y = \frac{y_0 \sqrt{x_0^2-1}}{\sqrt{x^2-1}}, \quad \forall x \in (1, \infty).$$

EXERCISES

(5) Solve the following initial value problems.

$$a) \begin{cases} y' - 2y = x e^{-2x} \\ y(0) = y_0 \end{cases}$$

$$b) \begin{cases} x y' - 2y = x^4 \\ y(1) = y_0 \end{cases}$$

$$c) \begin{cases} y' + y \tan x = \sin(2x) \\ y(0) = y_0 \end{cases}$$

$$d) \begin{cases} y' - (\cot x) y = 3x \sin x \\ y(\pi/4) = y_0 \end{cases}$$

$$e) \begin{cases} x y' + y = 3x^3 - 1 \\ y(1) = y_0 \end{cases}$$

$$f) \begin{cases} y' + e^x y = 2e^x \\ y(0) = y_0 \end{cases}$$

$$g) \begin{cases} y' + 2xy = x \exp(-x^2) \\ y(0) = y_0 \end{cases}$$

(6) Consider the initial value problem

$$\begin{cases} y' - 2xy = 1 \\ y(0) = y_0 \end{cases}$$

Show that its unique solution is:

$$y(x) = \exp(x^2) \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(x) + y_0 \right]$$

with $\operatorname{erf}(t)$ the error function defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$$

⑦ Bernoulli equations

A Bernoulli ordinary differential equation is an equation of the form

$$y' + p(x)y = q(x)y^n$$

with $n \in \mathbb{R}$.

a) Show that the substitution $u = y^{1-n}$ reduces the Bernoulli equation to a linear ordinary differential equation of the form

$$u' + (1-n)p(x)u = (1-n)q(x)$$

b) Use this substitution to solve the following Bernoulli initial value problem.

$$\begin{cases} y' + xy = xy^2 \\ y(0) = y_0 \end{cases}$$