Lecture Notes on Calculus 2

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Trigonometric identities

Trigonometric identities

$\boxed{a \pm b}$ $\Downarrow$

$$\left.\begin{aligned}
&\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a \\
&\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \\
&\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b} \\
&\cot(a \pm b) = \frac{\cot a \cot b \mp 1}{\cot b \pm \cot a} \quad \textbf{(!!)}
\end{aligned}\right\} \Longrightarrow$$

$\boxed{2a}$ $\Downarrow$

$$\begin{aligned}
&\sin(2a) = 2 \sin a \cos a \\
&\cos(2a) = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a \\
&\tan(2a) = \frac{2\tan a}{1 - \tan^2 a} \\
&\cot(2a) = \frac{\cot^2 a - 1}{2 \cot a}
\end{aligned}$$

- $\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$
- $\cos(a+b)\cos(a-b) = \cos^2 a - \sin^2 b$

$$\boxed{3a} \Longrightarrow \quad \begin{aligned} &\sin(3a) = -4\sin^3 a + 3\sin a \\ &\cos(3a) = +4\cos^3 a - 3\cos a \end{aligned} \qquad \tan(3a) = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}$$

In terms of

$\boxed{\cos 2a}$ $\Downarrow$

$$\sin^2 a = \frac{1 - \cos(2a)}{2} \qquad \cos^2 a = \frac{1 + \cos(2a)}{2}$$
$$\tan^2 a = \frac{1 - \cos(2a)}{1 + \cos(2a)} \qquad \cot^2 a = \frac{1 + \cos(2a)}{1 - \cos(2a)}$$

$\boxed{\tan(a/2)}$ $\Downarrow$

$$\sin a = \frac{2\tan(a/2)}{1 + \tan^2(a/2)} \qquad \cos a = \frac{1 - \tan^2(a/2)}{1 + \tan^2(a/2)}$$
$$\tan a = \frac{2\tan(a/2)}{1 - \tan^2(a/2)} \qquad \cot a = \frac{1 - \tan^2(a/2)}{2\tan(a/2)}$$

Transformation to

$\boxed{\text{sum}}$ $\Downarrow$

$$\left.\begin{aligned}
&2\sin a \cos b = \sin(a-b) + \sin(a+b) \\
&2\cos a \cos b = \cos(a-b) + \cos(a+b) \\
&2\sin a \sin b = \cos(a-b) - \cos(a+b)
\end{aligned}\right\} \Longrightarrow$$

$\boxed{\text{product}}$ $\Downarrow$

$$\begin{aligned}
&\sin a \pm \sin b = 2\sin\frac{a \pm b}{2}\cos\frac{a \mp b}{2} \\
&\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2} \\
&\cos a - \cos b = 2\sin\frac{a+b}{2}\sin\frac{b-a}{2} \quad \textbf{(!!)} \\
&\tan a \pm \tan b = \frac{\sin(a \pm b)}{\cos a \cos b} \\
&\cot a \pm \cot b = \frac{\sin(b \mp a)}{\sin a \sin b} \quad \textbf{(!!)}
\end{aligned}$$

Also note the factorizations:

- $1 \pm \sin a = \sin(\pi/2) \pm \sin a = 2\sin\dfrac{(\pi/2) \pm a}{2}\cos\dfrac{(\pi/2) \mp a}{2}$
- $\sin a \pm \cos b = \sin a \pm \sin(\pi/2 - b) = 2\sin\dfrac{a \pm (\pi/2 - b)}{2}\cos\dfrac{a \mp (\pi/2 - b)}{2}$
- $1 + \cos a = 2\cos^2(a/2)$
- $1 - \cos a = 2\sin^2(a/2)$

CAL 2.1: Integrals – Calculus 1 review

INTEGRALS - REVIEW FROM CALCULUS I

We summarize the Riemann definition of the integral and the main theorems from Calculus 1.

Riemann definition of the integral

Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and $[a,b] \subseteq A$.
We define:

$$x_k = a + (b-a)(k/n), \quad \forall k \in [n] \cup \{0\}$$

$$m_k(f|a,b,n) = \min_{x \in [x_{k-1}, x_k]} f(x), \quad \forall k \in [n]$$

$$M_k(f|a,b,n) = \max_{x \in [x_{k-1}, x_k]} f(x), \quad \forall k \in [n]$$

$$L_n(f|a,b) = \sum_{k=1}^{n} m_k(f|a,b,n)(x_k - x_{k-1})$$

$$U_n(f|a,b) = \sum_{k=1}^{n} M_k(f|a,b,n)(x_k - x_{k-1})$$

We say that:

f integrable on $[a,b] \iff \exists \ell \in \mathbb{R} : \lim_{n \in \mathbb{N}^*} L_n(f|a,b) = \lim_{n \in \mathbb{N}^*} U_n(f|a,b) = \ell$

If f is integrable on $[a,b]$, then we define the definite integral as follows:

$$\int_a^b f(x)dx = \lim_{n\in\mathbb{N}^*} L_n(f|a,b) = \lim_{n\in\mathbb{N}^*} U_n(f|a,b)$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \quad \text{and} \quad \int_a^a f(x)dx = 0$$

Basic properties of integrals

1) f continuous on $[a,b] \Rightarrow f$ integrable on $[a,b]$

2) Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and assume that f integrable on $[a,b]$. Then

$$\int_a^b f(x)dx = \lim_{n\in\mathbb{N}^*} \left[\frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right) \right]$$

3) Let f, g be integrable on $[a,b]$. Then:

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\forall \lambda \in \mathbb{R}: \int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$$

$$\left.\begin{array}{l} \forall k \in [n]: f_k \text{ integrable on } [a,b] \\ \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \int_a^b \left[\sum_{k=1}^{n} \lambda_k f_k(x) \right] dx = \sum_{k=1}^{n} \lambda_k \int_a^b f(x)dx$$

3) Charles theorem

$$\left.\begin{array}{l} f \text{ integrable at } I \\ I \text{ interval} \\ a,b,\gamma \in I \end{array}\right\} \Rightarrow \int_a^b f(x)dx = \int_a^\gamma f(x)dx + \int_\gamma^b f(x)dx$$

By induction, Charles theorem can be generalized to give:

$$\left.\begin{array}{l} f \text{ integrable on } I \\ I \text{ interval} \\ a,b,\gamma_1,\gamma_2,\dots,\gamma_n \in I \end{array}\right\} \Rightarrow \int_a^b f(x)dx = \int_a^{\gamma_1} f(x)dx + \sum_{k=1}^{n-1} \int_{\gamma_k}^{\gamma_{k+1}} f(x)dx + \int_{\gamma_n}^b f(x)dx$$

4) Integral bounding / comparison

$$\left.\begin{array}{l} f \text{ integrable on } [a,b] \\ \forall x \in [a,b]: f(x) \geq m \end{array}\right\} \Rightarrow \int_a^b f(x)dx \geq m(b-a)$$

$$\left.\begin{array}{l} f \text{ integrable on } [a,b] \\ \forall x \in [a,b]: f(x) \leq m \end{array}\right\} \Rightarrow \int_a^b f(x)dx \leq m(b-a)$$

← Note the implicit requirement that $a<b$!!

↳ For $m=0$, the above statement gives:

$$\left.\begin{array}{l} f \text{ integrable on } [a,b] \\ \forall x \in [a,b]: f(x) \geq 0 \end{array}\right\} \Rightarrow \int_a^b f(x)dx \geq 0$$

and an immediate corollary is that:

$$\left.\begin{array}{l} f,g \text{ integrable at } [a,b] \\ \forall x \in [a,b]: f(x) \geq g(x) \end{array}\right\} \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

5) Integral Mean-Value theorem

$$\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ g \text{ integrable on } [a,b] \\ \forall x \in [a,b]: g(x) \geq 0 \end{array}\right\} \Rightarrow \exists \xi \in [a,b]: \int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$

6) Absolute value of integral

$$\left.\begin{array}{l} f \text{ integrable on } [a,b] \\ |f| \text{ integrable on } [a,b] \end{array}\right\} \Rightarrow \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Fundamental theorem of calculus

1) Fundamental Theorem. I.

$$\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ \forall x \in [a,b]: F(x) = \int_a^x f(t)\,dt \end{array}\right\} \Rightarrow \left\{\begin{array}{l} F \text{ differentiable on } [a,b] \\ \forall x \in [a,b]: F'(x) = f(x) \end{array}\right.$$

Immediate consequences:

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x)$$

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\,b'(x) - f(a(x))\,a'(x)$$

2) Fundamental theorem II

$$\left.\begin{array}{l} F \text{ differentiable on } [a,b] \\ \forall x \in [a,b]: F'(x) = f(x) \\ f \text{ continuous on } [a,b] \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$$

↳ The FTC II motivates the definition of the indefinite integral.

$$\int f(x)\,dx = F(x) + c \Leftrightarrow \forall x \in \text{dom}(F): F'(x) = f(x)$$

F is the antiderivative of f.

3) Method of substitution theorem

$$\left.\begin{array}{l} g \text{ differentiable on } [a,b] \\ g' \text{ continuous on } [a,b] \\ f \text{ continuous on } g([a,b]) \end{array}\right\} \Rightarrow \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(g)\,dg$$

↳ Basic Integration formulas

1) $$\int x^a dx = \begin{cases} \frac{x^{a+1}}{a+1} + c & , \text{ if } a \neq -1 \\ \ln|x| + c & , \text{ if } a = -1 \end{cases}$$

2) $$\int \sin(x)\,dx = -\cos x + c \longrightarrow \int \sin(ax+b)\,dx = \frac{-1}{a}\cos(ax+b) + c$$

3) $$\int \cos(x)\,dx = \sin x + c \longrightarrow \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

4) $$\int \frac{dx}{\cos^2 x} = \tan x + c \longrightarrow \int \frac{dx}{\cos^2(ax+b)} = \frac{1}{a}\tan(ax+b) + c$$

5) $$\int \frac{dx}{\sin^2 x} = -\cot x + c \longrightarrow \int \frac{dx}{\sin^2(ax+b)} = \frac{-1}{a}\cot(ax+b) + c$$

6) $$\int a^x dx = \frac{a^x}{\ln a} + c \longrightarrow \int a^{bx} dx = \frac{a^{bx}}{b \ln a} + c$$

7) $$\int \frac{dx}{1+x^2} = \text{Arctan}(x) + c \longrightarrow \int \frac{dx}{x^2+a^2} = \frac{1}{a}\text{Arctan}\left(\frac{x}{a}\right) + c$$

8) $$\int \frac{dx}{\sqrt{1-x^2}} = \text{Arcsin}(x) + c \longrightarrow \int \frac{dx}{\sqrt{a^2-x^2}} = \text{Arcsin}\left(\frac{x}{|a|}\right) + c$$

CAL 2.2: Techniques of Integration

TECHNIQUES OF INTEGRATION

General Remarks

(1) During the evaluation of an indefinite integral, if we use two different techniques, it is possible to arrive to different results. These results will be correct if they are different only by a constant.

EXAMPLE

$$I = \int (x+1)\,dx = \frac{x^2}{2} + x + c$$

2nd method: Let $y = x+1 \Rightarrow dy = dx$, thus

$$I = \int y\,dy = \frac{y^2}{2} + c = \frac{(x+1)^2}{2} + c$$

$$= \frac{x^2+2x+1}{2} + c = \frac{x^2}{2} + x + \frac{1}{2} + c$$

(2) For integrals of the form $\rightarrow I = \int \frac{f'(x)}{f(x)}\,dx$

We use $\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c}$

EXAMPLE

$$I = \int \frac{4x-3}{2x^2-3x+1}\,dx = \int \frac{(2x^2-3x+1)'}{2x^2-3x+1}\,dx =$$
$$= \ln|2x^2-3x+1| + c$$

(3) For integrals of the form $I = \int \frac{f'(x)}{\sqrt{f(x)}}\,dx$ we use the identity

$$\boxed{\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c}$$

EXAMPLE

$$I = \int_0^1 \frac{3x}{\sqrt{x^2+3}}\,dx = \frac{3}{2}\int_0^1 \frac{2x}{\sqrt{x^2+3}}\,dx =$$
$$= \frac{3}{2}\int_0^1 \frac{(x^2+3)'}{\sqrt{x^2+3}}\,dx = \frac{3}{2}\left[2\sqrt{x^2+3}\right]_0^1 =$$
$$= 3\left[\sqrt{x^2+3}\right]_0^1 = 3\sqrt{1^2+3} - 3\sqrt{0^2+3} =$$
$$= 3(\sqrt{4}-\sqrt{3}) = 3(2-\sqrt{3}).$$

EXERCISES

(1) Evaluate the following integrals

a) $I = \int \frac{2x+3}{x^2+3x+5}\,dx$

b) $I = \int_1^2 \frac{3x^2}{x^3+5}\,dx$

c) $I = \int_1^3 \frac{8x^3+2}{x^4+x}\,dx$

d) $I = \int_2^3 \frac{2x}{\sqrt{x^2+1}}\,dx$

e) $I = \int_{\sqrt{2}}^3 \frac{x^2}{\sqrt{x^3-1}}\,dx$

f) $I = \int_2^{\sqrt{5}} \frac{5x^4+5}{\sqrt{x^5+5x+1}}\,dx$

g) $I = \int \frac{2x+1}{x^2+1}\,dx$

h) $I = \int \frac{3x-1}{\sqrt{1-x^2}}\,dx$

(2) Use the method of substitution to evaluate the following integrals.

a) $I = \int_0^{\pi/3} 3^{\sin x}\cos x\,dx$

b) $I = \int_0^{\pi/4} \frac{\cos x}{1+\sin^2 x}\,dx$

c) $I = \int_{-\ln 4}^{-1} \frac{3e^{x+1}}{\sqrt{1-e^{2x}}}\,dx$

d) $I = \int \frac{x^2 \sin(x^3-\pi)}{\cos^3(x^3-\pi)}\,dx$

e) $I = \int \frac{\sin(\ln(2x^3))}{x}\,dx$

③ Show that:

a) $\frac{1}{\cos x} = \frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x}$

b) $\int \frac{dx}{\cos x} = \ln\left|\frac{1}{\cos x} + \tan x\right| + C$

(Hint: Use (a) to show (b))

Integration by parts

This method is based on the following theorem:

Thm : Assume that
a) f, g differentiable at $[a, b]$
b) f', g' continuous at $[a, b]$
Then

$$\boxed{\int_a^b f'(x)\, g(x)\, dx = \Big[f(x) g(x) \Big]_a^b - \int_a^b f(x)\, g'(x)\, dx}$$

Proof

$$\int_a^b f'(x) g(x)\, dx + \int_a^b f(x) g'(x)\, dx =$$

$$= \int_a^b \left[f'(x) g(x) + f(x) g'(x) \right] dx =$$

$$= \int_a^b \left[f(x) g(x) \right]' dx = \Big[f(x)\, g(x) \Big]_a^b \Rightarrow$$

$$\Rightarrow \int_a^b f'(x)\, g(x) dx = \Big[f(x) g(x) \Big]_a^b - \int_a^b f(x)\, g'(x) dx \quad \square$$

(1) → $\boxed{I = \int P(x)\, e^{ax+b}\, dx}$

with $P(x)$ a polynomial.

We write $e^{ax+b} = \frac{1}{a}(e^{ax+b})'$ and apply integration by parts.

EXAMPLE

$$I = \int_0^1 (2x-3)\, e^{3x}\, dx = \frac{1}{3}\int_0^1 (2x-3)(e^{3x})'\, dx$$

$$= \frac{1}{3}\Big[(2x-3)e^{3x}\Big]_0^1 - \frac{1}{3}\int_0^1 (2x-3)'\, e^{3x}\, dx$$

$$= \frac{1}{3}\Big[(2\cdot 1-3)e^{3\cdot 1} - (2\cdot 0-3)e^{3\cdot 0}\Big] - \frac{1}{3}\int_0^1 2e^{3x}\, dx$$

$$= \frac{1}{3}\Big[-e^3 - (-3)\cdot 1\Big] - \frac{2}{3}\Big[\frac{e^{3x}}{3}\Big]_0^1 =$$

$$= \frac{3-e^3}{3} - \frac{2}{3}\,\frac{e^3-e^0}{3} =$$

$$= \frac{3-e^3}{3} - \frac{2(e^3-1)}{9} = \frac{3(3-e^3)-2(e^3-1)}{9} =$$

$$= \frac{9-3e^3-2e^3+2}{9} = \frac{11-5e^3}{9}$$

(2) →

$$I = \int P(x)\sin(ax+b)\,dx$$

$$I = \int P(x)\cos(ax+b)\,dx$$

with $P(x)$ a polynomial.

We use: $\sin(ax+b) = \frac{-1}{a}[\cos(ax+b)]'$

$$\cos(ax+b) = \frac{1}{a}[\sin(ax+b)]'$$

and apply integration by parts.

EXAMPLE

$$I = \int_0^{\pi/2} (2x-1)\cos(3x)\,dx = \frac{1}{3}\int_0^{\pi/2} (2x-1)(\sin(3x))'\,dx =$$

$$= \frac{1}{3}\left[(2x-1)\sin(3x)\right]_0^{\pi/2} - \frac{1}{3}\int_0^{\pi/2} (2x-1)'\sin(3x)\,dx =$$

$$= \frac{1}{3}\left[\left(2\frac{\pi}{2}-1\right)\sin\left(\frac{3\pi}{2}\right) - (2\cdot 0-1)\sin 0\right] - \frac{1}{3}\int_0^{\pi/2} 2\sin(3x)\,dx$$

$$= \frac{1}{3}\left[(\pi-1)\cdot(-1)-0\right] - \frac{2}{3}\left[\frac{-\cos(3x)}{3}\right]_0^{\pi/2} =$$

$$= \frac{1-\pi}{3} + \frac{2}{9}\left[\cos(3\pi/2) - \cos(0)\right]$$

$$= \frac{1-\pi}{3} + \frac{2}{9}(0-1) = \frac{1-\pi}{3} - \frac{2}{9} =$$

$$= \frac{3(1-\pi)-2}{9} = \frac{3-3\pi-2}{9} = \frac{1-3\pi}{9}$$

(3) →

$$I = \int e^{ax+b} \sin(cx+d)\,dx$$

$$I = \int e^{ax+b} \cos(cx+d)\,dx$$

Apply integration by parts twice. This leads to an equation with I unknown, which we then solve for I.

EXAMPLE

$$I = \int_0^{\pi/2} e^{-x}\sin(2x)\,dx = -\int_0^{\pi/2} (e^{-x})'\sin(2x)\,dx =$$

$$= -\left[e^{-x}\sin(2x)\right]_0^{\pi/2} + \int_0^{\pi/2} e^{-x}\cdot(\sin(2x))'\,dx$$

$$= -\left[e^{-\pi/2}\sin(2\cdot(\pi/2)) - e^0\sin 0\right] + 2\int_0^{\pi/2} e^{-x}\cos(2x)\,dx$$

$$= -e^{-\pi/2}\sin(\pi) - 2\int_0^{\pi/2} (e^{-x})'\cos(2x)\,dx =$$

$$= -e^{-\pi/2}\cdot 0 - 2\left[e^{-x}\cos(2x)\right]_0^{\pi/2} + 2\int_0^{\pi/2} e^{-x}\cdot(\cos(2x))'\,dx$$

$$= -2\left[e^{-\pi/2}\cos(2\cdot\pi/2) - e^0\cos 0\right] - 4\int_0^{\pi/2} e^{-x}\sin(2x)\,dx$$

$$= -2\left[e^{-\pi/2}\cos(\pi) - 1\right] - 4I = -2\left[e^{-\pi/2}\cdot(-1) - 1\right] - 4I$$

$$= 2e^{\pi/2} + 2 - 4I \Leftrightarrow I + 4I = 2e^{-\pi/2} + 2 \Leftrightarrow$$

$$\Leftrightarrow 5I = 2(e^{-\pi/2} + 1) \Leftrightarrow I = \frac{2(e^{-\pi/2}+1)}{5}$$

(4) → $\boxed{I = \int P(x) \ln(f(x))\, dx}$

Where $P(x)$ is a polynomial, and $f(x)$ is a function. The logarithm can be eliminated by writing $P(x) = Q'(x)$ and applying integration by parts:

EXAMPLE

a) $I = \int_1^2 (2x+1) \ln x \, dx = \int_1^2 (x^2+x)' \ln x \, dx =$

$= \left[(x^2+x) \ln x \right]_1^2 - \int_1^2 (x^2+x)(\ln x)' \, dx =$

$= \left[(2^2+2) \ln 2 - (1^2+1) \ln 1 \right] - \int_1^2 (x^2+x) \frac{1}{x} \, dx$

$= 6 \ln 2 - 2 \ln 1 - \int_1^2 (x+1) \, dx =$

$= 6 \ln 2 - \left[\frac{x^2}{2} + x \right]_1^2 = 6 \ln 2 - \left[\frac{2^2 - 1^2}{2} + (2-1) \right]$

$= 6 \ln 2 - \frac{3}{2} - 1 = 6 \ln 2 - \frac{5}{2}$

B) $$I=\int_0^{1/2}\ln\left(\frac{1-x}{1+x}\right)dx=\int_0^{1/2}(x)'\ln\left(\frac{1-x}{1+x}\right)dx$$

$$=\left[x\ln\left(\frac{1-x}{1+x}\right)\right]_0^{1/2}-\int_0^{1/2}x\left[\ln\left(\frac{1-x}{1+x}\right)\right]'dx \qquad (1)$$

We note that

$$\left[x\ln\left(\frac{1-x}{1+x}\right)\right]_0^{1/2}=\frac{1}{2}\ln\left(\frac{1-1/2}{1+1/2}\right)-0\ln\left(\frac{1-0}{1+0}\right)$$

$$=\frac{1}{2}\ln\left(\frac{1/2}{3/2}\right)=\frac{1}{2}\ln(1/3)=$$

$$=\frac{-\ln 3}{2} \qquad (2)$$

and

$$\left[\ln\left(\frac{1-x}{1+x}\right)\right]'=\frac{\left(\frac{1-x}{1+x}\right)'}{\frac{1-x}{1+x}}=$$

$$=\frac{1+x}{1-x}\,\frac{(1-x)'(1+x)-(1-x)(1+x)'}{(1+x)^2}=$$

$$=\frac{1+x}{1-x}\,\frac{(-1)(1+x)-(+1)(1-x)}{(1+x)^2}=$$

$$=\frac{1+x}{1-x}\,\frac{-1-x-1+x}{(1+x)^2}=\frac{1+x}{1-x}\,\frac{-2}{(1+x)^2}=$$

$$=\frac{-2}{(1-x)(1+x)}=\frac{2}{x^2-1} \qquad (3)$$

From Eq.(1), Eq.(2), Eq.(3), it follows that

$$I = \frac{-\ln 3}{2} - \int_0^{1/2} x \frac{2}{x^2-1}\,dx =$$

$$= \frac{-\ln 3}{2} - \int_0^{1/2} \frac{2x}{x^2-1}\,dx =$$

$$= \frac{-\ln 3}{2} - \int_0^{1/2} \frac{(x^2-1)'}{x^2-1}\,dx =$$

$$= \frac{-\ln 3}{2} - \Big[\ln|x^2-1|\Big]_0^{1/2} =$$

$$= \frac{-\ln 3}{2} - \Big[\ln|(1/2)^2-1| - \ln|0^2-1|\Big] =$$

$$= \frac{-\ln 3}{2} - \Big(\ln(3/4) - \ln 1\Big) =$$

$$= \frac{-\ln 3}{2} - (\ln 3 - \ln 4) =$$

$$= \left(-\frac{1}{2} - 1\right)\ln 3 + 2\ln 2$$

$$= \frac{-3\ln 3}{2} + 2\ln 2 = \frac{4\ln 2 - 3\ln 3}{2}$$

Miscelleneous Integrals

Sometimes we get integrals that cannot be categorized and human creativity becomes necessary

EXAMPLE

a) $$I = \int_0^{\pi/4} \frac{x}{\cos^2 x}\,dx = \int_0^{\pi/4} x\,(\tan x)'\,dx =$$

$$= \left[x\tan x\right]_0^{\pi/4} - \int_0^{\pi/4} (x)'\tan x\,dx =$$

$$= \left[\frac{\pi}{4}\tan\left(\frac{\pi}{4}\right) - 0\tan 0\right] - \int_0^{\pi/4} \tan x\,dx =$$

$$= \frac{\pi}{4} - \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx = \frac{\pi}{4} + \int_0^{\pi/4} \frac{(\cos x)'}{\cos x}\,dx =$$

$$= \frac{\pi}{4} + \left[\ln|\cos x|\right]_0^{\pi/4} = \frac{\pi}{4} + \left[\ln|\cos(\pi/4)| - \ln|\cos 0|\right]$$

$$= \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} + \ln\left(2^{-1/2}\right) =$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

b) $I = \int_0^1 \operatorname{Arctan} x \, dx = \int_0^1 (x)' \operatorname{Arctan} x \, dx =$

$= [x \operatorname{Arctan} x]_0^1 - \int_0^1 x (\operatorname{Arctan} x)' dx =$

$= 1 \operatorname{Arctan} 1 - 0 \cdot \cancelto{0}{\operatorname{Arctan} 0} - \int_0^1 x \cdot \frac{1}{1+x^2} dx$

$= \frac{\pi}{4} - \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} dx =$

$= \frac{\pi}{4} - \frac{1}{2} \int_0^1 \frac{(1+x^2)'}{(1+x^2)} dx = \frac{\pi}{4} - \frac{1}{2} \left[\ln |1+x^2| \right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2} \left[\ln(1+1^2) - \ln(\cancelto{0}{1+0^2}) \right] =$

$= \frac{\pi}{4} - \frac{1}{2} \ln 2$

EXERCISES

(4) Evaluate the following integrals

a) $I = \int_0^2 (3x+1)e^{2x}\,dx$

b) $I = \int_1^3 x^2 e^{-x}\,dx$

c) $I = \int_0^3 x(x+1)e^{4x}\,dx$

d) $I = \int_1^{\sqrt{2}} (x-1)(x^2+x+1)e^{-2x}\,dx$

e) $I = \int_0^{\pi/4} x^2 \cos(3x)\,dx$

f) $I = \int_{\pi/6}^{\pi/4} (2x-1)\sin(2x)\,dx$

g) $I = \int x^3 \sin x \cos x\,dx$

h) $I = \int_0^{\pi/6} x^2[\cos x \cos 2x - \sin x \sin 2x]\,dx$

(5) Evaluate the following integrals.

a) $I = \int_0^{\pi/4} e^{-x}\sin(2x)\,dx$

b) $I = \int e^{2x}\cos^2 x\,dx$

c) $I = \int_0^{\pi/6} e^{-3x}\sin^2 x\,dx$

d) $I = \int \frac{\ln x}{x^2}\,dx$

e) $I = \int_1^3 \frac{\ln(3x^2)}{x^5}\,dx$

f) $I = \int (x^2-x)\ln(3x+1)\,dx$

g) $I = \int x^4 \ln(x^7)\,dx$

h) $I = \int \ln\left(\frac{2x-1}{2x+1}\right)dx$

i) $I = \int_1^{\sqrt{2}} \ln\left(\frac{3x-2}{3x+2}\right)dx$

(6) Use integration by parts to show that

a) $\int e^{ax}\sin(bx)\,dx = \dfrac{e^{ax}(a\sin(bx)-b\cos(bx))}{a^2+b^2} + C$

b) $\int e^{ax}\cos(bx)\,dx = \dfrac{e^{ax}(a\cos(bx)+b\sin(bx))}{a^2+b^2} + C$

c) $\int x^a \ln x\,dx = \dfrac{x^{a+1}[(a+1)\ln x - 1]}{(a+1)^2} + C$, with $a \neq -1$.

d) $\int x^a (\ln x)^2\,dx = \dfrac{x^{a+1}[(a+1)^2(\ln x)^2 - 2(a+1)\ln x + 2]}{(a+1)^3} + C$, with $a \neq -1$.

(7) Evaluate the following integrals.

a) $I = \displaystyle\int_{\pi/6}^{\pi/3} \frac{2x+1}{\sin^2 x}\,dx$

b) $I = \displaystyle\int_0^{\sqrt{3}} x\,\mathrm{Arctan}(x)\,dx$

c) $I = \displaystyle\int_0^{\sqrt{2}/2} \mathrm{Arcsin}(x)\,dx$

d) $I = \displaystyle\int_1^e \ln^3(x).\,dx$

(8) We define the logarithmic integral function

$$Li(x) = \int_2^x \frac{dt}{\ln t}, \quad \forall x \in (1,+\infty)$$

and the incomplete gamma function

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt, \quad \forall x \in \mathbb{R},$$

Show that:

a) $I = \int_{e}^{a} \ln(\ln(x))\,dx = a\ln(\ln a) - [\mathrm{Li}(a) - \mathrm{Li}(e)]$

b) $I = \int_{2}^{a} \mathrm{Li}(x)\,dx = a\,\mathrm{Li}(a) + 2[\gamma(0, 2\ln a) - \gamma(0, 2\ln 2)]$

↳ Hint: In both cases, use integration by parts. Question (b) will also require the Fundamental theorem of Calculus – Part I and a change of variables.

Evaluating integrals via recursion

An expedient method for evaluating integrals is by deriving and employing recurrence relations as follows:

EXAMPLE

Define $I_n = \int_0^{\pi/2} \sin^n x\,dx$. Show that

a) $I_n = \frac{n-1}{n} I_{n-2}$, $\forall n \in \mathbb{N} : n \geq 3$

b) Evaluate $\int_0^{\pi/2} \sin^7 x\,dx$

Solution

a) $I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^{n-1} x \cdot \sin x\,dx =$

$= \int_0^{\pi/2} \sin^{n-1} x \cdot (-\cos x)'\,dx =$

$= \left[-\sin^{n-1} x \cos x\right]_0^{\pi/2} - \int_0^{\pi/2} (\sin^{n-1} x)' (-\cos x)\,dx$

$= \left[-\sin^{n-1}(\pi/2)\cos(\pi/2) + \sin^{n-1}(0)\cos 0\right]$

$+ \int_0^{\pi/2} (n-1)\sin^{n-2} x \cdot (\sin x)' \cos x\,dx =$

$$= \int_0^{\pi/2} (n-1)\sin^{n-2}x\cos^2x\,dx =$$

$$= \int_0^{\pi/2} (n-1)\sin^{n-2}x\,(1-\sin^2x)\,dx =$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2}x\,dx - (n-1)\int_0^{\pi/2} \sin^n x\,dx =$$

$$= (n-1)I_{n-2} - (n-1)I_n \Leftrightarrow$$

$$\Leftrightarrow I_n + (n-1)I_n = (n-1)I_{n-2} \Leftrightarrow$$

$$\Leftrightarrow nI_n = (n-1)I_{n-2} \Leftrightarrow I_n = \frac{n-1}{n} I_{n-2}.$$

β) $$\int_0^{\pi/2} \sin^7 x\,dx = I_7 = \frac{7-1}{7} I_5 = \frac{6}{7} I_5 =$$

$$= \frac{6}{7}\frac{5-1}{5} I_3 = \frac{6}{7}\frac{4}{5}\frac{3-1}{3} I_1 =$$

$$= \frac{6\cdot4\cdot2}{7\cdot5\cdot3}\int_0^{\pi/2} \sin x\,dx = \frac{2\cdot4\cdot2}{7\cdot5}\Big[-\cos x\Big]_0^{\pi/2} =$$

$$= \frac{16}{35}\Big[-\cos(\pi/2) + \cos 0\Big] = \frac{16}{35}(-0+1)$$

$$= \frac{16}{35}.$$

EXERCISES

(9) Consider the function $f_n(x) = \int_0^x t^n e^t \, dt$, $\forall n \in \mathbb{N}$, $\forall x \in \mathbb{R}$

a) Show that

$f_n(x) = x^n e^x - n f_{n-1}(x)$, $\forall x \in \mathbb{R}$, $\forall n \in \mathbb{N}^*$

b) Use the recurrence relation to evaluate the integral

$I = \int_0^{\ln 3} (x^2+2x-1)(x+1) e^x \, dx$

↳ Hint: First write I in terms of $f_n(x)$. Then evaluate the corresponding terms via the recurrence relation.

(10) Consider the functions:

$f_n(x) = \int_0^x t^n \sin t \, dt$, $\forall n \in \mathbb{N}$, $\forall x \in \mathbb{R}$

$g_n(x) = \int_0^x t^n \cos t \, dt$, $\forall n \in \mathbb{N}$, $\forall x \in \mathbb{R}$

a) Show that:

$f_n(x) = -x^n \cos x + n g_{n-1}(x)$, $\forall n \in \mathbb{N}^*$, $\forall x \in \mathbb{R}$

$g_n(x) = x^n \sin x - n f_{n-1}(x)$, $\forall n \in \mathbb{N}^*$, $\forall x \in \mathbb{R}$

b) Use the recurrence relations to evaluate

$I_1 = \int_0^{\pi/4} x^3 (2x-1) \cos x \, dx$

$I_2 = \int_0^{\pi/6} x^2 (x^2+2) \sin x \, dx$

(11) Consider the function $f_n(x) = \int_1^x (\ln t)^n \, dt$, $\forall n \in \mathbb{N}$, $\forall x \in (0, +\infty)$

a) Show that:

$f_n(x) = x(\ln x)^n - n f_{n-1}(x)$, $\forall x \in (0, +\infty)$, $\forall n \in \mathbb{N}^*$

b) Use the recurrence relation to evaluate the integral

$I = \int_1^{e^3} \ln x \, [\ln x - 2]^2 \, dx$

(12) We define the sequence $I_n = \int_0^{\pi/2} e^x \cos^n x \, dx$

a) Show that

$$I_n = \frac{-1}{n^2+1} + \frac{n(n-1)}{n^2+1} I_{n-2}, \quad \forall n \in \mathbb{N} - \{0,1,2\}$$

b) Use the recurrence relation to evaluate the integral

$J = \int_0^{\pi/2} e^x \cos^5 x \, dx$

(13) Use integration by parts and the fundamental theorem of Calculus - Part 1, to show that

$$\int_0^x dt \int_0^t dz \, f(z) = \int_0^x (x-t) f(t) \, dt$$

(14) Let $f: A \to \mathbb{R}$ be a twice differentiable equation with $a, b \in A$ and $f'(a) = f'(b) = 0$. Show that

$$\int_a^b f''(t) \, f(t) \, dt \leq 0$$

(Hint: Rewrite the integral in terms of $[f'(t)]^2$)

Integrals of rational functions

These are integrals of the form

$$I = \int \frac{P(x)}{Q(x)}\, dx$$

where P, Q are polynomials. In general, these integrals can always be evaluated, depending on the factorization of the denominator $Q(x)$.

(1) → $Q(x)$ as a product of distinct linear factors.

Assume that $Q(x)$ can be written as

$$Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$$

with $a_1 \neq a_2 \neq a_3 \cdots \neq a_n \neq a_1$. We also assume that $\deg P(x) < \deg Q(x)$. We define $[n] = \{1, 2, 3, \ldots, n\}$. Then we can show that

$$f(x) = \frac{P(x)}{\prod_{k\in[n]} (x-a_k)} = \sum_{k\in[n]} \frac{A_k}{x-a_k}$$

with

$$\forall k \in [n]: \; A_k = \left. \frac{P(x)}{\prod_{l\in[n]-\{k\}} (x-a_l)} \right|_{x=a_k}$$

Mnemonic Rule : To find A_k, delete the factor $x - a_k$ from $f(x)$ and the set $x = a_k$.

↳ This technique is known as the Parāvartya-Sūtra method and has been known since ancient times.

The resulting integrals are calculated by the following integration formula:

$$\int \frac{dx}{x-a} = \ln|x-a| + C$$

EXAMPLE

Evaluate $I = \int_0^1 \frac{3x-2}{x^3+x^2-4x-4}\,dx$

Solution

We note that

$Q(x) = x^3+x^2-4x-4 = x^2(x+1) - 4(x+1) =$

$= (x+1)(x^2-4) = (x+1)(x-2)(x+2)$

and therefore

$$f(x) = \frac{3x-2}{(x-2)(x+2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x+1}$$

with

$$A = \left.\frac{3x-2}{(x+2)(x+1)}\right|_{x=2} = \frac{3\cdot 2-2}{(2+2)(2+1)} = \frac{6-2}{4\cdot 3} =$$

$$= \frac{4}{4\cdot 3} = \frac{1}{3}$$

$$B = \left.\frac{3x-2}{(x-2)(x+1)}\right|_{x=-2} = \frac{3(-2)-2}{[(-2)-2][(-2)+1]} =$$

$$= \frac{-6-2}{(-4)(-1)} = \frac{-8}{4} = -2$$

$$C = \left.\frac{3x-2}{(x-2)(x+2)}\right|_{x=-1} = \frac{3(-1)-2}{[(-1)-2][(-1)+2]} =$$

$$= \frac{-3-2}{(-3)(+1)} = \frac{-5}{-3} = \frac{5}{3}$$

and therefore,

$$I=\int_0^1 \frac{3x-2}{(x-2)(x+2)(x+1)}\,dx=$$

$$=\frac{1}{3}\int_0^1 \frac{dx}{x-2}-2\int_0^1 \frac{dx}{x+2}+\frac{5}{3}\int_0^1 \frac{dx}{x+1}=$$

$$=\frac{1}{3}\Big[\ln|x-2|\Big]_0^1-2\Big[\ln|x+2|\Big]_0^1+\frac{5}{3}\Big[\ln|x+1|\Big]_0^1$$

$$=(1/3)(\ln|1-2|-\ln|0-2|)-2(\ln|1+2|-\ln|0+2|)$$
$$+(5/3)(\ln|1+1|-\ln|0+1|)=$$

$$=(1/3)(\ln 1-\ln 2)-2(\ln 3-\ln 2)+(5/3)(\ln 2-\ln 1)$$

$$=(1/3)(-\ln 2)-2(\ln 3-\ln 2)+(5/3)\ln 2$$

$$=[-(1/3)+2+(5/3)]\ln 2-2\ln 3$$

$$=((-1+6+5)/3)\ln 2-2\ln 3$$

$$=(10/3)\ln 2-2\ln 3.$$

EXERCISES

(16) Evaluate the following integrals:

a) $I = \int \frac{3x+1}{x^2+2x}\,dx$

b) $I = \int \frac{3x-2}{x^2+5x+6}\,dx$

c) $I = \int \frac{x^2+3x-1}{x^3-4x^2-4x+16}\,dx$

d) $I = \int \frac{2x^2+5x-1}{(x-1)(x-2)(x+3)}\,dx$

e) $I = \int \frac{x^2+4}{x^3-9x^2}\,dx$

f) $I = \int \frac{2x^2-x-1}{x^3-4x}\,dx$

g) $I = \int \frac{x^2+3x+5}{x^3-9x}\,dx$

h) $I = \int \frac{x^3+x^2-x+2}{3x^4-4x^3-12x^2+16}\,dx$

(17) Similarly for the following more challenging integrals.

a) $I = \int \frac{2x^2+3x-1}{x^3+7x^2+14x+8}\,dx$

b) $I = \int \frac{x^2+3x+8}{x^3+7x^2+7x-15}\,dx$

↳ To factor the denominator, use polynomial division as well as the results:

a) If $P(x)$ is a polynomial then:

$(x-a) \mid P(x) \Leftrightarrow P(a)=0$

b) If $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ then

$(P(p)=0 \wedge p \in \mathbb{Q}) \Rightarrow p \mid a_0$.

c) $I = \int \frac{2x^2+3}{(x^2-1)(x^2+1)}\,dx$

↳ Because the integral is exclusively dependent on x^2, we can use the Parāvartya-Sutra method to expand:

$$\frac{2y+3}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$$

and then set $y = x^2$.

(2) → $Q(x)$ is a product of linear factors some of which are repeated.

In general, it can be shown that

$$f(x) = \frac{P(x)}{(x-a)^n Q(x)} = \sum_{k=1}^{n} \frac{A_k}{(x-a)^k} + \frac{P_1(x)}{Q(x)}$$

if we assume that $\deg P(x) < n + \deg Q(x)$.
The highest order A_n can be found by:

$$\boxed{A_n = \left.\frac{P(x)}{Q(x)}\right|_{x=a} = \frac{P(a)}{Q(a)}}$$

The technique of the previous case can still be applied on all coefficients associated with linear non-repeating factors. There are multiple techniques for finding the coefficients $A_1, A_2, \dots, A_{n-1}$:

• In general:

$$\boxed{\forall k \in [n]: \quad A_k = \frac{1}{(n-k)!} \frac{d^{n-k}}{dx^{n-k}} \left(\frac{P(x)}{Q(x)}\right)\Bigg|_{x=a}}$$

will give all the coefficients, but calculating the derivatives can be a lot of work.
We define $0! = 1$ and $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$

• 2 For the special case $n=2$, we first find A_2. To find A_1, we multiply both sides of the decomposition with x and take the limit $x \to +\infty$. Consider

$$f(x) = \frac{P(x)}{(x-a)^2 Q(x)} = \frac{A_2}{(x-a)^2} + \frac{A_1}{(x-a)} + \frac{P_1(x)}{Q(x)}$$

Then, it follows that

$$\lim_{x\to+\infty} \frac{xP(x)}{(x-a)^2 Q(x)} = \lim_{x\to+\infty} \frac{A_2 x}{(x-a)^2} + \lim_{x\to+\infty} \frac{A_1 x}{x-a} + \lim_{x\to+\infty} \frac{xP_1(x)}{Q(x)} \iff$$

$$\iff L_1 = 0 + A_1 + L_2 \iff A_1 = L_1 - L_2$$

with

$$L_1 = \lim_{x\to+\infty} \frac{xP(x)}{(x-a)^2 Q(x)} \quad \text{and} \quad L_2 = \lim_{x\to+\infty} \frac{xP_1(x)}{Q(x)}$$

Note that A_1 can be calculated without knowing A_2! For $n>2$, this technique can still be used to find A_1. However, it will not work conveniently on the other coefficients. So, for $n=3$ we can easily find A_1, A_3 but need to do something else to find A_2. Same consideration applies for $n = 4, 5, 6, \ldots$

• 3 The above techniques can be augmented by subtracting the known terms in the partial fraction decomposition from the original function $f(x)$ and then attempting to decompose the resulting difference to obtain subleading terms.

EXAMPLE

Evaluate $I = \int \frac{3x^2 - 16x + 21}{(x-1)^2 (x+3)} \, dx$

Solution

We note that

$$f(x) = \frac{3x^2 - 16x + 21}{(x-1)^2 (x+3)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+3}$$

with

$$A = \left. \frac{3x^2 - 16x + 21}{x+3} \right|_{x=1} = \frac{3 \cdot 1^2 - 16 \cdot 1 + 21}{1+3} = \frac{3 - 16 + 21}{4} =$$

$$= \frac{8}{4} = 2$$

$$C = \left. \frac{3x^2 - 16x + 21}{(x-1)^2} \right|_{x=-3} = \frac{3(-3)^2 - 16(-3) + 21}{((-3)-1)^2} =$$

$$= \frac{27 + 48 + 21}{16} = \frac{96}{16} = 6$$

$$B = \frac{1}{(2-1)!} \frac{d}{dx} \left(\frac{3x^2 - 16x + 21}{x+3} \right) \Bigg|_{x=1} =$$

$$= \left. \frac{(3x^2 - 16x + 21)'(x+3) - (3x^2 - 16x + 21)(x+3)'}{(x+3)^2} \right|_{x=1} =$$

$$= \left. \frac{(6x - 16)(x+3) - (3x^2 - 16x + 21)}{(x+3)^2} \right|_{x=1}$$

$$= \frac{(6 \cdot 1 - 16)(1+3) - (3 \cdot 1^2 - 16 \cdot 1 + 21)}{(1+3)^2} =$$

$$= \frac{(6-16)4-(3-16+21)}{4^2} = \frac{(-10)4-8}{16} =$$

$$= \frac{-40-8}{16} = \frac{-48}{16} = -3$$

and therefore

$$I = \int \frac{3x^2-16x+21}{(x-1)^2(x+3)}\,dx =$$

$$= \int \left[\frac{2}{(x-1)^2} + \frac{-3}{x-1} + \frac{6}{x+3} \right] dx =$$

$$= \frac{2}{-1}\,\frac{1}{x-1} - 3\ln|x-1| + 6\ln|x+3| + C$$

$$= \frac{-2}{x-1} - 3\ln|x-1| + 6\ln|x+3| + C$$

↳ Alternate methods for determining B.

2nd method: Given A, C:

$$\frac{3x^2-16x+21}{(x-1)^2(x+3)} = \frac{2}{(x-1)^2} + \frac{B}{x-1} + \frac{6}{x+3} \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} \frac{x(3x^2-16x+21)}{(x-1)^2(x+3)} = \lim_{x\to+\infty} \frac{2x}{(x-1)^2} + \lim_{x\to+\infty} \frac{Bx}{x-1} + \lim_{x\to+\infty} \frac{6x}{x+3} \Rightarrow$$

$$\Rightarrow 3 = 0 + B + 6 \Rightarrow B = 3-6 = -3.$$

3rd method : Given A, G we have

$$\frac{3x^2-16x+21}{(x-1)^2(x+3)} = \frac{2}{(x-1)^2} + \frac{B}{x-1} + \frac{6}{x+3} \Rightarrow$$

$$\Rightarrow \frac{B}{x-1} = \frac{3x^2-16x+21}{(x-1)^2(x+3)} - \frac{2}{(x-1)^2} - \frac{6}{x+3} =$$

$$= \frac{(3x^2-16x+21) - 2(x+3) - 6(x-1)^2}{(x-1)^2(x+3)} =$$

$$= \frac{3x^2-16x+21-2x-6-6(x^2-2x+1)}{(x-1)^2(x+3)}$$

$$= \frac{3x^2-16x+21-2x-6-6x^2+12x-6}{(x-1)^2(x+3)}$$

$$= \frac{(3-6)x^2+(-16-2+12)x+(21-6-6)}{(x-1)^2(x+3)}$$

$$= \frac{-3x^2-6x+9}{(x-1)^2(x+3)} = \frac{-3(x^2+2x-3)}{(x-1)^2(x+3)} =$$

$$= \frac{-3(x-1)(x+3)}{(x-1)^2(x+3)} = \frac{-3}{x-1} \Rightarrow B = -3.$$

EXERCISES

(18) Evaluate the following integrals.

a) $I = \int \frac{x^2+3x-1}{(x-2)^2(x+1)}\,dx$ B) $I = \int \frac{2x^2+7x-1}{(x+1)^2(2x-1)}\,dx$

c) $I = \int \frac{x^3+3x+1}{(x+2)^3(x-1)}\,dx$

↳ To find the partial function decomposition, we note that

$$R(x) = \frac{x^3+3x+1}{(x+2)^3(x-1)} = \frac{A}{(x+2)^3} + \frac{B}{(x+2)^2} + \frac{C}{x+2} + \frac{D}{x-1}$$

and find A, D via Parāvartya-Sūtra method and then simplify:

$$R(x) - \frac{A}{(x+2)^3} - \frac{D}{x-1} = \ldots = \frac{B_1x+C_1}{(x+2)^2}$$

which does not really require further decomposition.

d) $I = \int \frac{x^2+3x+5}{(3x+2)^2(x-1)}\,dx$

↳ Rewrite $(3x+2)^2 = 9(x+2/3)^2$ and then apply the Parāvartya-Sutra method.

e) $I = \int \frac{x^3+5x+1}{(x-4)^2(x-2)^2}\,dx$

↳ To find the partial fraction decomposition, we note that

$$R(x) = \frac{x^3+5x+1}{(x-4)^2(x-2)^2} = \frac{A}{(x-4)^2} + \frac{B}{x-4} + \frac{C}{(x-2)^2} + \frac{D}{x-2}$$

and use the Parāvartya-Sūtra method to find A, C.
We the simplify

$$R(x) - \frac{A}{(x-4)^2} - \frac{C}{(x-2)^2} = \ldots = \frac{B}{x-4} + \frac{D}{x-2}$$

and use a second iteration of the Parāvartya-Sūtra method to find B, D.

f) $$I = \int \frac{3x^3+2x^2+5}{(x+1)^2(x+3)^2 - (x+1)(x+3)^3}\,dx$$

↳ You need to factor the denominator in order to find the partial fraction decomposition and evaluate the integral.

③ → **Integrals of the form** $I = \int \frac{Ax+B}{ax^2+bx+c}\,dx$ **with** ax^2+bx+c **an irreducible quadratic** (i.e. cannot be factored into a product of real linear factors).

To evaluate such integrals, we break the function to a contribution of the form

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

and a contribution of the form

$$\int \frac{dx}{ax^2+bx+c}$$

To evaluate the second integral, we complete the square on the denominator, using the identity

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}$$

with $\Delta = b^2-4ac$, and use the identities

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{Arctan}\left(\frac{x}{a}\right) + C$$

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx$$

to evaluate the integral.

EXAMPLE

Evaluate $I = \int_{-1}^{1} \frac{3x+2}{x^2+2x+6}\, dx$

Solution

We note that

$$I = \int_{-1}^{1} \frac{3x+2}{x^2+2x+6}\, dx = \int_{-1}^{1} \frac{(3/2)(2x+2)-3+2}{x^2+2x+6}\, dx =$$

$$= \frac{3}{2}\int_{-1}^{1} \frac{2x+2}{x^2+2x+6}\, dx - \int_{-1}^{1} \frac{dx}{x^2+2x+6}$$

$$= (3/2) I_1 - I_2$$

with

$$I_1 = \int_{-1}^{1} \frac{2x+2}{x^2+2x+6}\, dx = \int_{-1}^{1} \frac{(x^2+2x+6)'}{x^2+2x+6}\, dx =$$

$$= \left[\ln|x^2+2x+6| \right]_{-1}^{1} =$$

$$= \ln|1^2+2\cdot 1+6| - \ln|(-1)^2+2(-1)+6| =$$

$$= \ln|1+2+6| - \ln|1-2+6| = \ln 9 - \ln 5$$

and

$$I_2 = \int_{-1}^{1} \frac{dx}{x^2+2x+6} = \int_{-1}^{1} \frac{dx}{(x^2+2x+1)+5} =$$

$$= \int_{-1}^{1} \frac{dx}{(x+1)^2+5} = \int_{0}^{2} \frac{dx}{x^2+5} = \int_{0}^{2} \frac{dx}{x^2+(\sqrt{5})^2}$$

$$= \left[\frac{1}{\sqrt{5}} \operatorname{Arctan}\left(\frac{x}{\sqrt{5}}\right) \right]_{0}^{2} =$$

$$= \frac{1}{\sqrt{5}} \operatorname{Arctan}\left(\frac{2}{\sqrt{5}}\right) - \frac{1}{\sqrt{5}} \operatorname{Arctan}\left(\frac{0}{\sqrt{5}}\right) =$$

$$= \frac{1}{\sqrt{5}} \operatorname{Arctan}\left(\frac{2}{\sqrt{5}}\right)$$

and therefore

$$I = \frac{3I_1}{2} - I_2 = \frac{3(\ln 9 - \ln 5)}{2} - \frac{1}{\sqrt{5}} \operatorname{Arctan}\left(\frac{2}{\sqrt{5}}\right)$$

EXERCISES

⑤ Evaluate the following integrals.

a) $I = \int_1^4 \frac{dx}{x+3}$

b) $I = \int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{x-1}$

c) $I = \int_3^{3\sqrt{3}} \frac{dx}{x^2+9}$

d) $I = \int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{x^2+2}$

e) $I = \int_4^5 \frac{3x\,dx}{x^2+1}$

f) $I = \int_{\sqrt{2}}^{\sqrt{5}} \frac{5x\,dx}{x^2+3}$

g) $I = \int_{\sqrt{3}}^{2} \frac{5x\,dx}{x^2-4}$

h) $I = \int_1^{1+\sqrt{2}} \frac{x\sqrt{2}}{x^2+2}\,dx$

i) $I = \int \frac{2x+3}{x^2+3x+1}\,dx$

j) $I = \int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{2x-1}{x^2-x+3}\,dx$

k) $I = \int \frac{3x+4}{x^2+5x+2}\,dx$

l) $I = \int \frac{5x-2}{x^2+4x+3}\,dx$

m) $I = \int_1^3 \frac{x+1}{x^2+2x+3}\,dx$

n) $I = \int_{\sqrt{3}}^{\sqrt{7}} \frac{x\sqrt{2}+3}{x^2+7x+12}\,dx$

o) $I = \int \frac{(\sqrt{2}+\sqrt{3})x+1}{x^2+(\sqrt{2}+1)x+\sqrt{2}}\,dx$

p) $I = \int_1^{\sqrt{3}} \frac{x\sqrt{2}+\sqrt{3}}{x^2\sqrt{2}+(\sqrt{3}+1)x+2}\,dx$

q) $I = \int \frac{5-x}{x^2-x(\pi+4)+4\pi}\,dx$

(4) → Integrals with non-repeating quadratic factors

Consider the integral

$$I = \int \frac{P(x)}{(x^2+ax+b)Q(x)}\,dx$$

with $\deg P(x) < 2 + \deg Q(x)$, where we assume that $Q(x)$ does not contain any additional factors x^2+ax+b. Then, the corresponding decomposition is

$$f(x) = \frac{Ax+B}{x^2+ax+b} + \frac{P_1(x)}{Q(x)}$$

If $z = p+qi$ is a zero of the equation $x^2+ax+b=0$ then we can calculate A, B via the equation

$$\boxed{A(p+qi)+B = \left.\frac{P(x)}{Q(x)}\right|_{x=p+qi}}$$

This reduces to a system of two equations via the property

$$\forall a_1, a_2, b_1, b_2 \in \mathbb{R}: (a_1+b_1 i = a_2+b_2 i \Leftrightarrow a_1 = a_2 \wedge b_1 = b_2)$$

that we can use to solve for A, B.

EXAMPLE

Evaluate $I = \int \frac{3x}{(x^2+2)(x-1)}\,dx$

Solution

We note that

$$f(x) = \frac{3x}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1}$$

with

$$C = \left.\frac{3x}{x^2+2}\right|_{x=1} = \frac{3\cdot 1}{1^2+2} = \frac{3}{3} = 1$$

$$A(i\sqrt{2})+B = \left.\frac{3x}{x-1}\right|_{x=i\sqrt{2}} = \frac{3\sqrt{2}i}{i\sqrt{2}-1} =$$

$$= \frac{3\sqrt{2}i\,(i\sqrt{2}+1)}{(i\sqrt{2}-1)(i\sqrt{2}+1)} = \frac{6i^2+3\sqrt{2}i}{2i^2-1} =$$

$$= \frac{-6+3\sqrt{2}i}{-2-1} = \frac{-6+3\sqrt{2}i}{-3} = 2-\sqrt{2}i \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} A\sqrt{2} = -\sqrt{2} \\ B = 2 \end{cases} \Leftrightarrow \begin{cases} A = -1 \\ B = 2 \end{cases}$$

and therefore

$$I = \int \frac{3x}{(x^2+2)(x-1)}\,dx =$$

$$= \int \left[\frac{-x+2}{x^2+2} + \frac{1}{x-1}\right] dx =$$

$$= \frac{-1}{2}\int \frac{2x\,dx}{x^2+2} + 2\int \frac{dx}{x^2+2} + \int \frac{dx}{x-1} =$$

$$= \frac{-1}{2}\int \frac{(x^2+2)'}{x^2+2}\,dx + 2\int \frac{dx}{x^2+(\sqrt{2})^2} + \ln|x-1| + C$$

$$= \frac{-1}{2}\ln|x^2+2| + 2\,\frac{1}{\sqrt{2}}\,\text{Arctan}\left(\frac{x}{\sqrt{2}}\right) + \ln|x-1| + C$$

$$= \sqrt{2}\,\text{Arctan}(x/\sqrt{2}) - (1/2)\ln(x^2+2) + \ln|x-1| + C$$

(19) Evaluate the following integrals.

a) $I = \int \frac{x^2-x+1}{(x^2+2x+4)(x+3)}\,dx$ b) $I = \int \frac{3x+1}{(x^2-x+1)(x+3)}\,dx$

c) $I = \int \frac{3x+1}{2x^3+16}\,dx$ d) $I = \int \frac{x^2+2x-1}{x^3-27}\,dx$

↳ For (c) and (d) the denominators can be factored using

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

e) $I = \int \frac{3x}{x^4-1}\,dx$ f) $I = \int \frac{dx}{2x^4-32}$

↳ Note the factorization

$$x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

g) $I = \int \frac{3x-2}{x^4-2x^3-x+2}\,dx$ h) $I = \int \frac{x^2-3x+1}{x^3+3x^2+4x+12}\,dx$

(5) → For all other more complex cases, such as
(a) Repeated quadratic factors
(b) More than one linear repeating factor
we can use the <u>method of undetermined coefficients</u>

- 1 Rewrite the partial fraction decomposition as a polynomial equation.
- 2 Set x equal to the zeroes of the denominators factors to obtain the coefficients of the leading terms of the partial fraction decomposition
- 3 The coefficients of the subleading terms can be obtained by trying other strategically chosen values of x.

EXAMPLE

$$I = \int \frac{6x^2 - 15x + 22}{(x+3)(x^2+2)^2}\, dx$$

Solution

Let $f(x) = \frac{6x^2 - 15x + 22}{(x+3)(x^2+2)^2} = \frac{A}{x+3} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2} \Leftrightarrow$

$\Leftrightarrow A(x^2+2)^2 + (Bx+C)(x+3)(x^2+2) + (Dx+E)(x+3) = 6x^2 - 15x + 22, \forall x \in \mathbb{R}$ (1)

▶ We begin by trying the zeroes of the denominator:

For $x = -3$, from Eq. (1):

$A[(-3)^2+2]^2 + 0 + 0 = 6(-3)^2 - 15(-3) + 22 \Leftrightarrow$

$\Leftrightarrow A(9+2)^2 = 6\cdot 9 + 45 + 22 \Leftrightarrow 121A = 54+45+22 \Leftrightarrow$

$\Leftrightarrow 121A = 121 \Leftrightarrow A = 1.$

For $x = i\sqrt{2}$ (the root of x^2+2), from Eq.(1)

$0+0+[D(i\sqrt{2})+E](i\sqrt{2}+3) = 6(i\sqrt{2})^2 - 15(i\sqrt{2}) + 22 \Leftrightarrow$

$\Leftrightarrow D(i\sqrt{2})+E = \frac{6(i\sqrt{2})^2 - 15(i\sqrt{2}) + 22}{i\sqrt{2}+3} =$

$= \frac{6(-2) - (15\sqrt{2})i + 22}{i\sqrt{2}+3} = \frac{-12 - (15\sqrt{2})i + 22}{3+i\sqrt{2}}$

$= \frac{10 - (15\sqrt{2})i}{3+i\sqrt{2}} = \frac{5[2-(3\sqrt{2})i]}{3+i\sqrt{2}}$

$= \frac{5[2-(3\sqrt{2})i][3-i\sqrt{2}]}{[3+i\sqrt{2}][3-i\sqrt{2}]} = \frac{5[6-(2\sqrt{2})i-(9\sqrt{2})i-6]}{9-(-2)}$

$= \frac{5[-(11\sqrt{2})i]}{11} = -(5\sqrt{2})i \Leftrightarrow$

$\Leftrightarrow \begin{cases} D\sqrt{2} = -5\sqrt{2} \\ E = 0 \end{cases} \Leftrightarrow \begin{cases} D = -5 \\ E = 0 \end{cases}$

↳ Note that we may indeed assume $D, E \in \mathbb{R}$ and then use the proposition:

$\forall a,b,c,d \in \mathbb{R} : (a+bi = c+di \Leftrightarrow (a=c \wedge b=d))$

to justify the above argument. Also note that using just one of the two conjugate roots is sufficient for obtaining both coefficients.

↳ Before dealing with the coefficients of the subleading terms, we use $A=1 \wedge D=-5 \wedge E=0$ to simplify Eq.(1)

$(1) \Leftrightarrow (x+2)^2 + (Bx+C)(x+3)(x^2+2) + (-5x)(x+3) = 6x^2 - 15x + 22, \forall x \in \mathbb{R}$.

To find $B, C \in \mathbb{R}$ we try:

For $x = 0$; we have:

$(0+2)^2 + (0+C)(0+3)(0+2) + 0 = 0 - 0 + 22 \Leftrightarrow$

$\Leftrightarrow 4 + 6C = 22 \Leftrightarrow 6C = 22 - 4 = 18 \Leftrightarrow C = 3$

For $x = 1$, we have:

$(1^2+2)^2 + (B+C)(1+3)(1^2+2) + (-5 \cdot 1)(1+3) = 6 \cdot 1^2 - 15 \cdot 1 + 22 \Leftrightarrow$

$\Leftrightarrow 9 + 12(B+C) - 20 = 6 - 15 + 22 \Leftrightarrow 12(B+C) - 11 = 13 \Leftrightarrow$

$\Leftrightarrow 12(B+C) = 24 \Leftrightarrow B + C = 2 \Leftrightarrow B + 3 = 2 \Leftrightarrow B = -1.$

To summarize: $(A, B, C, D, E) = (1, -1, 3, -5, 0)$

and therefore:

$$f(x) = \frac{1}{x+3} + \frac{-x+3}{x^2+2} + \frac{-5x}{(x^2+2)^2}$$

Since

$$I_1 = \int \frac{dx}{x+3} = \ln|x+3| + C$$

$$I_2 = \int \frac{-x+3}{x^2+2}\,dx = \int \frac{-x\,dx}{x^2+2} + \int \frac{3\,dx}{x^2+2} =$$

$$= \frac{-1}{2}\int \frac{2x\,dx}{x^2+2} + 3\int \frac{dx}{x^2+2} =$$

$$= \frac{-1}{2}\int \frac{(x^2+2)'}{x^2+2}\,dx + 3\int \frac{dx}{x^2+(\sqrt{2})^2} =$$

$$= \frac{-1}{2}\ln|x^2+2| + \frac{3}{\sqrt{2}}\operatorname{Arctan}\left(\frac{x}{\sqrt{2}}\right)$$

and

$$I_3 = \int \frac{-5x\,dx}{(x^2+2)^2} = \frac{-5}{2}\int \frac{2x\,dx}{(x^2+2)^2} = \frac{-5}{2}\int \frac{(x^2+2)'\,dx}{(x^2+2)^2} =$$

$$= \frac{-5}{2}\int\left[\frac{-1}{x^2+2}\right]'dx = \frac{-5}{2}\,\frac{-1}{x^2+2} + C =$$

$$= \frac{5}{2(x^2+2)} + C$$

and it follows that

$$I = I_1 + I_2 + I_3 =$$

$$= \ln|x+3| - \frac{1}{2}\ln|x^2+2| + \frac{3}{\sqrt{2}}\operatorname{Arctan}\left(\frac{x}{\sqrt{2}}\right) + \frac{5}{2(x^2+2)} + C.$$

↳ Note that if $E \neq 0$, the problem becomes much harder. However as long as the quadratic term goes up to power 2, this method is safe given the following method for calculating

$$I = \int \frac{dx}{(x^2+a^2)^2}$$

We use a clever rewrite and integration by parts:

$$I = \int \frac{dx}{(x^2+a^2)^2} = \frac{1}{a^2}\int \frac{x^2+a^2-x^2}{(x^2+a^2)^2}\,dx = \frac{1}{a^2}\int \frac{dx}{x^2+a^2} - \frac{1}{a^2}\int \frac{x^2\,dx}{(x^2+a^2)^2}$$

$$= \frac{1}{a^2}\,\frac{1}{a}\operatorname{Arctan}\left(\frac{x}{a}\right) + \frac{1}{a^2}\int \frac{x}{2}\,\frac{-2x}{(x^2+a^2)^2}\,dx =$$

$$= \frac{1}{a^3}\operatorname{Arctan}\left(\frac{x}{a}\right) + \frac{1}{a^2}\int \frac{x}{2}\left(\frac{1}{x^2+a^2}\right)'dx$$

$$= \frac{1}{a^3}\operatorname{Arctan}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} - \frac{1}{a^2}\int \left(\frac{x}{2}\right)'\frac{1}{x^2+a^2}\,dx =$$

$$= \frac{1}{a^3}\text{Arctan}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} - \frac{1}{2a^2}\int \frac{dx}{x^2+a^2} =$$

$$= \frac{1}{a^3}\text{Arctan}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} - \frac{1}{2a^2}\,\frac{1}{a}\text{Arctan}\left(\frac{x}{a}\right) + C$$

$$= \frac{1}{2a^3}\text{Arctan}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} + C$$

A second method to circumvent doing difficult integrals is to use the partial fraction decomposition:

$$f(x) = \frac{6x^2-15x+22}{(x+3)(x^2+2)^2} = \frac{A}{x+3} + \frac{B}{x+i\sqrt{2}} + \frac{C}{(x+i\sqrt{2})^2} + \frac{D}{x-i\sqrt{2}} + \frac{E}{(x-i\sqrt{2})}$$

however B, C, D, E may be complex numbers. Now, we may use Parāvartya-Sutrā method to find A, C, E and undetermined coefficients to find B, D. However the price you pay is that you need to use complex analysis theory to deal with the resulting integrals. Alternatively, you can use the partial fraction decomposition above and stay with real integrals at the expense of getting more challenging integrals.

EXERCISES

(20) Evaluate the following integrals.

a) $I = \int \frac{dx}{(x^2+4)^2}$ b) $I = \int_0^{\sqrt{3}} \frac{dx}{(x^2+3)^2}$

c) $I = \int \frac{(5x-2)}{(x^2+5)^2}\,dx$ d) $I = \int \frac{3x^4+2x^3+22x^2+10x+47}{(x+2)(x^2+3)^2}\,dx$

e) $I = \int \frac{8x^4-6x^3+90x^2-39x+243}{(x-2)(x^2+7)^2}\,dx$

f) $I = \int_0^1 \frac{dx}{x^4+1}$ g) $I = \int \frac{3x}{x^4+x^2+1}\,dx$

↳ Note the 4th-order factorization

$$x^4+a^4 = x^4+2a^2x^2+a^4-2a^2x^2 =$$
$$= (x^2+a^2)^2 - 2a^2x^2 =$$
$$= [(x^2+a^2)-(a\sqrt{2})x][(x^2+a^2)+(a\sqrt{2})x]$$
$$= (x^2-(a\sqrt{2})x+a^2)(x^2+(a\sqrt{2})x+a^2)$$

This approach can also be used to factor a biquadratic polynomial of the form ax^4+bx^2+c.

h) $I = \int \frac{x^3\,dx}{(x^2+3)^2}$ i) $I = \int \frac{dx}{x^6-64}$

↳ Note the 6th order factorization combining difference of squares and difference of cubes:

$$x^6-a^6 = (x^3-a^3)(x^3+a^3) =$$
$$= (x-a)(x^2+ax+a^2)(x+a)(x^2-ax+a^2)$$

⑥ → Integrals $I = \int \frac{P(x)}{Q(x)}\,dx$ with $\deg P(x) \geq \deg Q(x)$

Before doing partial fraction decomposition, we use long polynomial division to divide the numerator with the denominator

EXAMPLE

$$I = \int \frac{x^4}{x^2+x-2}\,dx$$

Solution

We divide: $x^4 / (x^2+x-2)$:

$$\begin{array}{l|l}
x^4+0x^3+0x^2+0x+0 & x^2+x-2 \\
\hline
-x^4-1x^3+2x^2 & x^2-x+3 \\
\hline
\quad -1x^3+2x^2+0x+0 & \\
\quad +1x^3+x^2-2x+0 & \\
\hline
\qquad 3x^2-2x+0 & \\
\qquad -3x^2-3x+6 & \\
\hline
\qquad\qquad -5x+6 &
\end{array}$$

thus:

$$f(x) = \frac{x^4}{x^2+x-2} = x^2-x+3+\frac{-5x+6}{x^2+x-2} =$$

$$= x^2-x+3+\frac{-5x+6}{(x+2)(x-1)} =$$

$$= x^2-x+3+\frac{A}{x+2}+\frac{B}{x-1}$$

with

$$A = \left.\frac{-5x+6}{x-1}\right|_{x=-2} = \frac{-5(-2)+6}{-2-1} = \frac{10+6}{-3} = \frac{-16}{3}$$

$$B = \left.\frac{-5x+6}{x+2}\right|_{x=1} = \frac{-5\cdot 1+6}{1+2} = \frac{1}{3}$$

and therefore

$$I = \int \frac{x^4}{x^2+x-2}\,dx = \int\left[x^2-x+3+\frac{-16/3}{x+2}+\frac{1/3}{x-1}\right]dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 3x - \frac{16\ln|x+2|}{3} + \frac{\ln|x-1|}{3} + C$$

EXERCISES

(21) Evaluate the following integrals:

a) $I = \int \frac{2x^3 + 3x^2 - 16}{x^2 - 4}\, dx$

b) $I = \int \frac{x^5 - 3x^4 - 2x^3 - 4x^2 + 2}{x^3 + x^2 + x + 1}\, dx$

c) $I = \int \frac{x^6 + 2x^4 - 8x^2 + 3x + 1}{x^2 + 4}\, dx$

d) $I = \int \frac{x^7 + 6x^5 + 7x^3 + x^2 - 5x + 3}{x^4 + 6x^2 + 9}\, dx$

e) $I = \int \frac{x^6 + 3x^5 + 8x^4 + 23x^3 + 9x^2 + 21x + 3}{x^3 + 3x^2 + 7x + 21}\, dx$

f) $I = \int \frac{x^6 - 3x^4 + 4x^3 - 3x^2 + 6x - 7}{(x^2 + 1)^2 - x^2}\, dx$

Integrals that reduce to rational integrals

1. Form $\boxed{I = \int P(x)\,\text{Arctan}(f(x))\,dx}$

We use integration by parts to eliminate the $\text{Arctan}(f(x))$ factor.

EXAMPLE

$$I = \int (2x+1)\,\text{Arctan}(x-1)\,dx$$

Solution

$$I = \int (2x+1)\,\text{Arctan}(x-1)\,dx = \int (x^2+x)'\,\text{Arctan}(x-1)\,dx =$$

$$= (x^2+x)\,\text{Arctan}(x-1) - \int (x^2+x)\,[\text{Arctan}(x-1)]'\,dx =$$

$$= (x^2+x)\,\text{Arctan}(x-1) - \int (x^2+x)\,\frac{(x-1)'}{(x-1)^2+1}\,dx =$$

$$= (x^2+x)\,\text{Arctan}(x-1) - \int \frac{x^2+x}{x^2-2x+1+1}\,dx$$

$$= (x^2+x)\,\text{Arctan}(x-1) - \int \frac{x^2+x}{x^2-2x+2}\,dx$$

$$= (x^2+x)\,\text{Arctan}(x-1) - I_1$$

Divide x^2+x with x^2-2x+2

x^2+x+0	x^2-2x+2
$-x^2+2x-2$	1
$3x-2$	

therefore

$$I_1=\int\frac{x^2+x}{x^2-2x+2}\,dx=\int\left[1+\frac{3x-2}{x^2-2x+2}\right]dx=$$

$$=\int\left[1+\frac{(3/2)(2x-2)+1}{x^2-2x+2}\right]dx=$$

$$=\int\left[1+\frac{3}{2}\frac{(x^2-2x+2)'}{x^2-2x+2}+\frac{1}{x^2-2x+2}\right]dx=$$

$$=x+\frac{3}{2}\ln|x^2-2x+2|+\int\frac{dx}{(x-1)^2+1}=$$

$$=x+\frac{3}{2}\ln|x^2-2x+2|+\operatorname{Arctan}(x-1)+C\Rightarrow$$

$$\Rightarrow I=(x^2+x)\operatorname{Arctan}(x-1)-I_1=$$

$$=(x^2+x)\operatorname{Arctan}(x-1)-[x+(3/2)\ln|x^2-2x+2|+\operatorname{Arctan}(x-1)]+C=$$

$$=(x^2+x-1)\operatorname{Arctan}(x-1)-x-(3/2)\ln|x^2-2x+2|+C$$

(2) → Form $\boxed{I = \int P(x) \ln|f(x)|\, dx}$

We use integration by parts to eliminate the $\ln|f(x)|$ factor. The resulting integral most generally will be a rational integral.

EXAMPLE

$$I = \int 2x \ln|x^2-1|\, dx$$

Solution

$$I = \int 2x \ln|x^2-1|\, dx = \int (x^2)' \ln|x^2-1|\, dx =$$

$$= x^2 \ln|x^2-1| - \int x^2 [\ln|x^2-1|]'\, dx =$$

$$= x^2 \ln|x^2-1| - \int x^2 \frac{(x^2-1)'}{x^2-1}\, dx =$$

$$= x^2 \ln|x^2-1| - \int x^2 \frac{2x}{x^2-1}\, dx =$$

$$= x^2 \ln|x^2-1| - \int \frac{2x^3}{x^2-1}\, dx \qquad (1)$$

We divide $2x^3 / (x^2-1)$

$$\begin{array}{r|l} 2x^3+0x^2+0x+0 & x^2-1 \\ \hline -2x^3+0x^2+2x+0 & 2x \\ \hline 2x & \end{array}$$

and note that

$$f(x) = \frac{2x^3}{x^2-1} = 2x + \frac{2x}{x^2-1} = 2x + \frac{(x^2-1)'}{x^2-1}$$

and therefore from Eq.(1):

$$I = x^2 \ln|x^2-1| - \int \left[2x + \frac{(x^2-1)'}{x^2-1} \right] dx =$$

$$= x^2 \ln|x^2-1| - [x^2 + \ln|x^2-1|] + C =$$

$$= x^2 \ln|x^2-1| - x^2 - \ln|x^2+1| + C =$$

$$= (x^2-1)\ln|x^2-1| - x^2 + C$$

(3) → Form: $\boxed{I=\int F\left(x,\sqrt[n]{ax+b}\right)dx}$

$$\boxed{I=\int F\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx}$$

Methodology:

- 1 Let $y=\sqrt[n]{ax+b}$ or $y=\sqrt[n]{(ax+b)/(cx+d)}$
- 2 Solve for x to obtain $x=g(y)$
- 3 Calculate the $dx=g'(y)dy$
- 4 It follows that

$$I=\int F(g(y),y)\,g'(y)dy$$

which is, in general, a rational integral.

EXAMPLE

$$I=\int_{-1/2}^{-1/3}\frac{1}{x}\sqrt{\frac{1+x}{1-x}}\,dx$$

Solution

Let $y=\sqrt{\frac{1+x}{1-x}} \Leftrightarrow y^2=\frac{1+x}{1-x} \Leftrightarrow y^2(1-x)=1+x \Leftrightarrow$

$\Leftrightarrow y^2-xy^2=1+x \Leftrightarrow x(1+y^2)=y^2-1 \Leftrightarrow x=\frac{y^2-1}{y^2+1}$ (1)

For $x=-1/2 \Rightarrow y=\sqrt{\frac{1-1/2}{1+1/2}}=\sqrt{\frac{1/2}{3/2}}=\sqrt{\frac{1}{3}}=\frac{\sqrt{3}}{3}$ (2)

For $x=-1/3 \Rightarrow y=\sqrt{\frac{1-1/3}{1+1/3}}=\sqrt{\frac{2/3}{4/3}}=\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}$ (3)

Also:

$$dx=\left(\frac{y^2-1}{y^2+1}\right)'dy=\frac{(y^2-1)'(y^2+1)-(y^2-1)(y^2+1)'}{(y^2+1)^2}dy=$$

$$=\frac{2y(y^2+1)-2y(y^2-1)}{(y^2+1)^2}dy=\frac{2y(y^2+1-y^2+1)}{(y^2+1)^2}dy$$

$$=\frac{4y}{(y^2+1)^2}dy \quad (4)$$

From Eq.(1), Eq.(2), Eq.(3), Eq.(4):

$$I=\int_{\sqrt{3}/3}^{\sqrt{2}/2}\frac{y^2+1}{y^2-1}\cdot y\cdot\frac{4y}{(y^2+1)^2}dy=$$

$$=\int_{\sqrt{3}/3}^{\sqrt{2}/2}\frac{4y^2}{(y^2-1)(y^2+1)}dy=\int_{\sqrt{3}/3}^{\sqrt{2}/2}\frac{4y^2}{(y-1)(y+1)(y^2+1)}dy$$

$$=\int_{\sqrt{3}/3}^{\sqrt{2}/2}\left[\frac{A}{y-1}+\frac{B}{y+1}+\frac{Cy+D}{y^2+1}\right]dy: \quad (5)$$

with

$$A=\left.\frac{4y^2}{(y+1)(y^2+1)}\right|_{y=1}=\frac{4\cdot 1^2}{(1+1)(1^2+1)}=\frac{4}{2\cdot 2}=1$$

$$B=\left.\frac{4y^2}{(y-1)(y^2+1)}\right|_{y=-1}=\frac{4(-1)^2}{((-1)-1)((-1)^2+1)}=\frac{4}{(-2)2}=-1$$

$$Ci+D=\left.\frac{4y^2}{(y-1)(y+1)}\right|_{y=i}=\frac{4i^2}{(i-1)(i+1)}=\frac{4(-1)}{i^2-1}=$$

$$=\frac{-4}{(-1)-1}=\frac{-4}{-2}=2 \Leftrightarrow \begin{cases} C=0 \\ D=2 \end{cases}$$

and from Eq. (5)

$$I = \int_{\sqrt{3}/3}^{\sqrt{2}/2} \left[\frac{1}{y-1} - \frac{1}{y+1} + \frac{2}{y^2+1} \right] dy =$$

$$= \left[\ln|y-1| - \ln|y+1| + 2\,\mathrm{Arctan}(y) \right]_{\sqrt{3}/3}^{\sqrt{2}/2} =$$

$$= \left[\ln|\sqrt{2}/2 - 1| - \ln|\sqrt{2}/2 + 1| + 2\,\mathrm{Arctan}(\sqrt{2}/2) \right] -$$
$$- \left[\ln|\sqrt{3}/3 - 1| - \ln|\sqrt{3}/3 + 1| + 2\,\mathrm{Arctan}(\sqrt{3}/3) \right]$$

$$= \ln \left| \frac{(\sqrt{2}/2 - 1)(\sqrt{3}/3 + 1)}{(\sqrt{2}/2 + 1)(\sqrt{3}/3 - 1)} \right| + 2 \left[\mathrm{Arctan}(\sqrt{2}/2) - \mathrm{Arctan}(\sqrt{3}/3) \right]$$

$$= \ln \left| \frac{(\sqrt{2} - 2)(\sqrt{3} + 3)}{(\sqrt{2} + 2)(\sqrt{3} - 3)} \right| + 2 \left[\mathrm{Arctan}(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln \left| \frac{(\sqrt{2} - 2)^2 (\sqrt{3} + 3)^2}{(\sqrt{2} + 2)(\sqrt{3} - 3)(\sqrt{2} - 2)(\sqrt{3} + 3)} \right| +$$
$$+ 2 \left[\mathrm{Arctan}(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln \left| \frac{(2 - 4\sqrt{2} + 4)(3 + 6\sqrt{3} + 9)}{(2 - 4)(3 - 9)} \right| + 2 \left[\mathrm{Arctan}(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln \left| \frac{(6 - 4\sqrt{2})(12 + 6\sqrt{3})}{(-2)(-6)} \right| + 2 \left[\mathrm{Arctan}(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln \left| \frac{2(3 - 2\sqrt{2})\,6(2 + \sqrt{3})}{(-2)(-6)} \right| + 2 \left[\mathrm{Arctan}(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln \left| (3 - 2\sqrt{2})(2 + \sqrt{3}) \right| + 2 \left[\mathrm{Arctan}(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln \left| 6 - 4\sqrt{2} + 3\sqrt{3} - 2\sqrt{6} \right| + 2 \left[\mathrm{Arctan}(\sqrt{2}/2) - \pi/6 \right]$$

EXERCISES

(22) Evaluate the following indefinite integrals:

a) $I = \int \text{Arctan}(x)\,dx$ b) $I = \int x\,\text{Arctan}(x)\,dx$

c) $I = \int x^2\,\text{Arctan}(x)\,dx$

(23) Use the indefinite integrals from the previous exercise to evaluate the integral

$$I = \int_0^{\sqrt{3}} (x+1)(x-2)\,\text{Arctan}(x)\,dx$$

(24) Evaluate the following integrals:

a) $I = \int \text{Arctan}(x^2)\,dx$ b) $I = \int x\,\text{Arctan}(x^2)\,dx$

c) $I = \int (3x-2)\ln x\,dx$ d) $I = \int 2x\ln(x^2-x+2)\,dx$

e) $I = \int x^2\ln(x^2+5x+6)\,dx$ f) $I = \int 2x\ln(x^4+1)\,dx$

g) $I = \int x\ln(x^6-1)\,dx$

(25) Evaluate the following integrals

a) $I = \int_2^3 x\sqrt{2x-3}\,dx$

b) $I = \int_0^4 \frac{x\,dx}{1+\sqrt{2x+1}}$

c) $I = \int \frac{2x-1}{\sqrt[3]{3x+2}}\,dx$

d) $I = \int \frac{x\,dx}{\sqrt[3]{x+1}+\sqrt{x+1}}$

d) $I = \int_{-1/2}^{1/2} \sqrt{\frac{1-x}{1+x}}\,dx$

e) $I = \int_0^1 \frac{x\,dx}{\sqrt[3]{2x+1}}$

f) $I = \int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$

f) $I = \int_1^2 \frac{dx}{\sqrt{2x+1}-\sqrt[4]{2x+1}}$

g) $I = \int_{-1}^1 x\sqrt{\frac{2-x}{2+x}}\,dx$

↳ Hint: For (d) use $y = \sqrt[6]{x+1}$ such that $\sqrt{x+1} = y^3$ and $\sqrt[3]{x+1} = y^2$.
Similar idea applies to (f)

Trigonometric Integrals

- Be sure to remove all trigonometric identities

① → Form:

$$I = \int f(\sin x)\cos x\,dx \quad \leftarrow y = \sin x$$

$$I = \int f(\cos x)\sin x\,dx \quad \leftarrow y = \cos x$$

EXAMPLES

a) $I = \int_0^{\pi/4} \frac{\sin x}{2+\cos x}\,dx$

Solution

Let $y = \cos x$. Then $dy = -\sin x\,dx \Rightarrow \sin x\,dx = -dy$.

For $x = 0$: $y = \cos 0 = 1$

For $x = \pi/4$: $y = \cos(\pi/4) = \sqrt{2}/2$

It follows that

$$I = \int_1^{\sqrt{2}/2} \frac{-dy}{2+y} = \left[-\ln|y+2|\right]_1^{\sqrt{2}/2} =$$

$$= [-\ln|\sqrt{2}/2+2|] - [-\ln|1+2|] =$$

$$= -\ln(2+\sqrt{2}/2) + \ln 3$$

B) $I = \int_0^{\pi/2} \sin^5 x \, dx$

Solution

$$I = \int_0^{\pi/2} \sin^5 x \, dx = \int_0^{\pi/2} \sin^4 x \sin x \, dx = \int_0^{\pi/2} (\sin^2 x)^2 \sin x \, dx$$

$$= \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x \, dx$$

Let $y = \cos x \Rightarrow dy = -\sin x \, dx \Rightarrow \sin x \, dx = -dy$

For $x=0$: $y = \cos 0 = 1$

For $x=\pi/2$: $y = \cos(\pi/2) = 0$

It follows that:

$$I = \int_1^0 (1-y^2)^2 (-1) dy = \int_0^1 (1-y^2)^2 dy =$$

$$= \int_0^1 (1-2y^2+y^4) dy = \left[y - \frac{2y^3}{3} + \frac{y^5}{5} \right]_0^1 =$$

$$= \left[1 - \frac{2 \cdot 1^3}{3} + \frac{1^5}{5} \right] - (0-0+0) =$$

$$= 1 - \frac{2}{3} + \frac{1}{5} = \frac{15-2\cdot 5+1\cdot 3}{15} = \frac{15-10+3}{15} =$$

$$= \frac{8}{15}$$

c) $I = \int \frac{dx}{\sin x}$

Solution

$$I = \int \frac{dx}{\sin x} = \int \frac{\sin x\, dx}{\sin^2 x} = \int \frac{\sin x\, dx}{1-\cos^2 x} =$$

$$= \int \frac{\sin x\, dx}{(1-\cos x)(1+\cos x)}$$

Let $y = \cos x \Rightarrow dy = -\sin x\, dx \Rightarrow \sin x\, dx = -dy$

Then,

$$I = \int \frac{-dy}{(1-y)(1+y)} = \int \frac{dy}{(y-1)(y+1)} = \int \left[\frac{A}{y-1} + \frac{B}{y+1}\right] dy$$

with

$$A = \left.\frac{1}{y+1}\right|_{y=1} = \frac{1}{1+1} = \frac{1}{2}$$

$$B = \left.\frac{1}{y-1}\right|_{y=-1} = \frac{1}{-1-1} = \frac{-1}{2}$$

and therefore

$$I = \int \left[\frac{1/2}{y-1} + \frac{-1/2}{y+1}\right] dy =$$

$$= (1/2)\ln|y-1| - (1/2)\ln|y+1| + C$$

$$= (1/2)\left[\ln|y-1| - \ln|y+1|\right] + C$$

$$= \frac{1}{2}\ln\left(\frac{|y-1|}{|y+1|}\right) = \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$$

d) $I = \int_0^{\pi/4} \tan^5 x\,dx$

Solution

$$I = \int_0^{\pi/4} \tan^5 x\,dx = \int_0^{\pi/4} \frac{\sin^5 x}{\cos^5 x}\,dx = \int_0^{\pi/4} \frac{\sin^4 x}{\cos^5 x}\sin x\,dx =$$

$$= \int_0^{\pi/4} \frac{(1-\cos^2 x)^2}{\cos^5 x}\sin x\,dx$$

Let $y = \cos x \Rightarrow dy = -\sin x\,dx \Rightarrow \sin x\,dx = -dy$

For $x = 0$: $y = \cos 0 = 1$

For $x = \pi/4$: $y = \cos(\pi/4) = \sqrt{2}/2$

It follows that

$$I = \int_1^{\sqrt{2}/2} \frac{(1-y^2)^2}{y^5}(-1)\,dy = \int_{\sqrt{2}/2}^1 \frac{1-2y^2+y^4}{y^5}\,dy =$$

$$= \int_{\sqrt{2}/2}^1 \left[y^{-5} - 2y^{-3} + y^{-1}\right]dy = \left[\frac{y^{-4}}{-4} - 2\frac{y^{-2}}{-2} + \ln|y|\right]_{\sqrt{2}/2}^1$$

$$= \left[\frac{-1}{4y^4} + \frac{1}{y^2} + \ln|y|\right]_{\sqrt{2}/2}^1 =$$

$$= \left[\frac{-1}{4\cdot 1^4} + \frac{1}{1^2} + \ln|1|\right] - \left[\frac{-1}{4(\sqrt{2}/2)^4} + \frac{1}{(\sqrt{2}/2)^2} + \ln(\sqrt{2}/2)\right]$$

$$= \left(1 - \frac{1}{4}\right) + \frac{1}{4(1/4)} - \frac{1}{1/2} - \ln 2^{-1/2} =$$

$$= 1 - \frac{1}{4} + 1 - 2 - (-1/2)\ln 2 = -\frac{1}{4} + \frac{1}{2}\ln 2$$

↳ Note that all of the integrals above follow the general form

$$I = \int \sin^a x \cos^b x \, dx$$

with $a, b \in \mathbb{Z}$ and $\boxed{a \text{ odd} \vee b \text{ odd}}$
Using the identity $\sin^2 x + \cos^2 x = 1$ we see that such integrals easily reduce to form 1.

EXERCISES

(26) Evaluate the following integrals:

a) $I = \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin^2 x}}\,dx$ b) $I = \int_0^{\pi/4} \frac{\sin x}{1+\cos^2 x}\,dx$

c) $I = \int_0^{\pi/4} \sin^3 x\,dx$ d) $I = \int_0^{\pi/3} \cos^3 x\,dx$

e) $I = \int \sin^7 x\,dx$ f) $I = \int \cos^9 x\,dx$

g) $I = \int \sin^2 x \cos^3 x\,dx$ h) $I = \int_{\pi/4}^{\pi/3} \cos^3 x \sin^4 x\,dx$

i) $I = \int \frac{dx}{\cos^3 x}$ j) $I = \int_{\pi/6}^{\pi/3} \frac{dx}{\sin^5 x}$

k) $I = \int \tan^3 x\,dx$ l) $I = \int_0^{\pi/4} \tan^5 x\,dx$

m) $I = \int_0^{\pi/3} \cot^3(x)\,dx$ n) $I = \int_0^{\pi/8} \cot^5(2x)\,dx$

o) $I = \int_{\pi/6}^{\pi/2} \sin(2x)\left[\cos^2 x - \sin^2 x\right]dx$

p) $I = \int \cos(3x)\left[\sin x \cos(2x) + \sin(2x)\cos x\right]dx$

↳ Integrals (o) and (p) can be reduced to form I by using the appropriate trig. identities for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$.

(2) → Form :

$I=\int \frac{f(\tan x)}{\cos^2 x}\,dx$	→ Let $y=\tan x$
$I=\int \frac{f(\cot x)}{\sin^2 x}\,dx$	→ Let $y=\cot x$
$I=\int f(\tan x)\,dx$	→ Let $y=\tan x$

EXAMPLES

a) $I=\int_{\pi/3}^{\pi/4} \frac{\ln(\tan x)}{\cos^2 x}\,dx$

Solution

Let $y=\tan x=g(x) \Rightarrow \begin{cases} dy=dx/\cos^2 x \\ g(\pi/3)=\tan(\pi/3)=\sqrt{3} \\ g(\pi/4)=\tan(\pi/4)=1 \end{cases} \rightarrow$

$$\Rightarrow I=\int_{\sqrt{3}}^{1} \ln y\,dy=\int_{\sqrt{3}}^{1} (y)'\ln y\,dy=$$

$$=[y\ln y]_{\sqrt{3}}^{1}-\int_{\sqrt{3}}^{1} y(\ln y)'\,dy=$$

$$=[1\ln 1-\sqrt{3}\ln(\sqrt{3})]-\int_{\sqrt{3}}^{1} y(1/y)\,dy=$$

$$=[0-\sqrt{3}\ln(\sqrt{3})]-\int_{\sqrt{3}}^{1} dy=-\sqrt{3}\ln\sqrt{3}-[y]_{\sqrt{3}}^{1}=$$

$$=-\sqrt{3}\ln(\sqrt{3})-[1-\sqrt{3}]=\sqrt{3}-1-\sqrt{3}\ln(\sqrt{3})=$$

$$=-1+\sqrt{3}(1-\ln(\sqrt{3}))$$

b) $I = \int_0^{\pi/3} \tan^4 x\,dx$

Solution

Let $y = \tan x = g(x) \Rightarrow dy = (1+\tan^2 x)\,dx = (1+y^2)\,dx \Rightarrow$

$$\Rightarrow dx = \frac{dy}{1+y^2}$$

and note that

$g(0) = \tan 0 = 0$

$g(\pi/3) = \tan(\pi/3) = \sqrt{3}$

It follows that $I = \int_0^{\sqrt{3}} \frac{y^4}{1+y^2}\,dy$

We divide y^4 by $1+y^2$:

$$\begin{array}{r|l} y^4+0y^3+0y^2+0y+0 & y^2+0y+1 \\ \hline -y^4+0y^3-y^2 & y^2-1 \\ \cline{1-1} -y^2+0y+0 & \\ +y^2+0y+1 & \\ \cline{1-1} 1 & \end{array}$$

and therefore:

$$I = \int_0^{\sqrt{3}} \frac{y^4}{1+y^2}\,dy = \int_0^{\sqrt{3}} \left[y^2 - 1 + \frac{1}{y^2+1} \right] dx =$$

$$= \left[\frac{y^3}{3} - y + \text{Arctan}(y) \right]_0^{\sqrt{3}} =$$

$$= \frac{(\sqrt{3})^3 - 0^3}{3} - [\sqrt{3} - 0] + [\text{Arctan}(\sqrt{3}) - \text{Arctan}(0)] =$$

$$= \frac{3\sqrt{3}}{3} - \sqrt{3} + \frac{\pi}{3} - 0 = \sqrt{3} - \sqrt{3} + \pi/3 = \pi/3.$$

EXERCISES

(27) Evaluate the following functions.

a) $I=\int_0^{\pi/4} \frac{\ln(1+\tan x)}{\cos^2 x}\,dx$ b) $I=\int_0^{\pi/6} \frac{\tan(2x)}{\cos^2 x}\,dx$

c) $I=\int \frac{\tan x+\tan 2x}{\cos^2(3x)[1-\tan x\tan(2x)]}\,dx$

↳ For (b) and (c) use the trigonometric identity for $\tan(a+b)$.

d) $I=\int_0^{\pi/4} \frac{\sin(\pi\tan x)}{\cos^2 x}\,dx$ e) $I=\int_0^{\pi/6} \tan^2 x\,dx$

f) $I=\int_0^{\pi/3} \tan^6 x\,dx$ g) $I=\int \frac{3dx}{\tan x+2}$

h) $I=\int_0^{\pi/4} \frac{\tan x}{1+\tan x}\,dx$ i) $I=\int \frac{dx}{\tan(1+\tan x)}$

j) $I=\int_0^{\pi/3} \frac{dx}{1+\tan^3 x}$ k) $I=\int \frac{dx}{1+\tan^4 x}$

(28) Let $I_n=\int_0^{\pi/4} \tan^n(x)\,dx$, $\forall n\in\mathbb{N}^*$. Show that:

$$I_n=\frac{1}{n-1}-I_{n-2},\quad \forall n\in\mathbb{N}-\{0,1,2\}$$

(29) Let $I_n = \int_0^{\pi/3} \frac{dx}{\cos^n x}$, $\forall n \in \mathbb{N}^*$.

a) Show that : $I_n = \frac{2^{n-2}\sqrt{3}}{n-1} + \frac{n-2}{n-1} I_{n-2}$, $\forall n \in \mathbb{N} - \{0,1,2\}$

b) Evaluate I_1 and use the recurrence relation to show that

$$\int_0^{\pi/3} \frac{dx}{\cos^3 x} = \sqrt{3} + \frac{1}{2} \ln\left(\frac{3}{2} + \sqrt{3}\right)$$

③ → Form : $\boxed{I = \int \frac{\tan^a x}{\cos^b x}\,dx}$ $\boxed{I = \int \frac{\cot^a x}{\sin^b x}\,dx}$

We distinguish between the following cases:

Case 1 : Assume that $a \in \mathbb{R}$ $\wedge$ b even integer

Use the identities

$\boxed{\frac{1}{\cos^2 x} = 1 + \tan^2 x}$ $\boxed{\frac{1}{\sin^2 x} = 1 + \cot^2 x}$

Then reduce integral to form 2:

$\boxed{\int \frac{f(\tan x)}{\cos^2 x}\,dx \text{ or } \int \frac{f(\cot x)}{\sin^2 x}\,dx}$

Case 2 : Assume that a odd integer $\wedge$ $b \in \mathbb{R}$.

Use the identity $\sin^2 x + \cos^2 x = 1$

Then reduce integral to form 1:

$\boxed{\int f(\cos x)\sin x\,dx \text{ or } \int f(\sin x)\cos x\,dx}$

Then integrals of form 1 or 2 can be evaluated as explained previously. The final result should be expressed using root notation whenever fractional powers are involved.

EXAMPLE

a) $I = \int \tan^3 x \sqrt{\cos x}\ dx$

Solution

$$I = \int \tan^3 x \sqrt{\cos x}\ dx = \int \frac{\sin^3 x}{\cos^3 x} \cos^{1/2} x\ dx =$$

$$= \int \frac{\sin^2 x \cos^{1/2} x}{\cos^3 x} \sin x\ dx =$$

$$= \int \frac{(1-\cos^2 x)\cos^{1/2} x}{\cos^3 x} \sin x\ dx$$

Let $y = \cos x \Rightarrow dy = -\sin x\, dx \Rightarrow \sin x\, dx = -dy$

and therefore,

$$I = \int \frac{(1-y^2) y^{1/2}}{y^3} (-1) dy = \int (y^{-3} - y^{-1}) y^{1/2} (-1)\ dy =$$

$$= \int (y^{-1} - y^{-3}) y^{1/2} dy = \int (y^{-1/2} - y^{-5/2})\ dy =$$

$$= \frac{y^{1/2}}{1/2} - \frac{y^{-3/2}}{-3/2} + C = 2\sqrt{y} + \frac{2}{3} \frac{1}{y\sqrt{y}} + C$$

$$= 2\sqrt{\cos x} + \frac{2}{3} \frac{1}{\cos x \sqrt{\cos x}} + C$$

b) $I = \int \frac{dx}{\cos^4 x \tan x \sqrt{\tan x}}$

Solution

$$I = \int \frac{dx}{\cos^4 x \tan x \sqrt{\tan x}} = \int \frac{\tan^{-3/2} x}{\cos^2 x} \frac{1}{\cos^2 x} dx =$$

$$= \int \frac{\tan^{-3/2} x (1+\tan^2 x)}{\cos^2 x} dx$$

Let $y = \tan x \Rightarrow dy = \frac{dx}{\cos^2 x}$ and therefore:

$$I = \int y^{-3/2} (1+y^2) dy = \int (y^{-3/2} + y^{1/2}) dy =$$

$$= \frac{y^{-1/2}}{-1/2} + \frac{y^{3/2}}{3/2} + C = \frac{-2}{\sqrt{y}} + \frac{2}{3} y\sqrt{y} + C =$$

$$= \frac{-2}{\sqrt{\tan x}} + \frac{2\tan x \sqrt{\tan x}}{3} + C$$

EXERCISES

(30) Evaluate the following integrals:

a) $I = \int \frac{dx}{\tan^3 x \cos^4 x}$

b) $I = \int_0^{\pi/6} \frac{\sqrt{\tan x}}{\cos^4 x}\, dx$

c) $I = \int \frac{\tan^3 x}{\cos^6 x}\, dx$

d) $I = \int_0^{\pi/3} \frac{\tan^5 x}{\sqrt{\cos x}}\, dx$

e) $I = \int \frac{\tan^3 x}{\cos x \sqrt{\cos x}}\, dx$

f) $I = \int_0^{\pi/4} \frac{\tan x \sqrt{\tan x}}{\cos^6 x}\, dx$

(4) → Form : Products of trig. functions

Products of trigonometric functions can be eliminated using the following identities:

$$\sin a \cdot \cos b = (1/2)[\sin(a-b) + \sin(a+b)]$$
$$\cos a \cos b = (1/2)[\cos(a-b) + \cos(a+b)]$$
$$\sin a \sin b = (1/2)[\cos(a-b) - \cos(a+b)]$$

EXAMPLE

a) $$I = \int_0^{\pi} \sin(3x)\cos(2x)\,dx =$$

$$= \int_0^{\pi} (1/2)[\sin(3x-2x) + \sin(3x+2x)]\,dx$$

$$= \int_0^{\pi} (1/2)[\sin x + \sin 5x]\,dx =$$

$$= \frac{1}{2}\left[-\cos x - \frac{\cos 5x}{5}\right]_0^{\pi} =$$

$$= \frac{1}{2}\left[-(\cos\pi - \cos 0) - \frac{\cos 5\pi - \cos 0}{5}\right]$$

$$= \frac{1}{2}\left[-((-1)-1) - \frac{(-1)-1}{5}\right] =$$

$$= \frac{1}{2}\left[1 + 1 + \frac{2}{5}\right] = \frac{1}{2}\,\frac{12}{5} = \frac{6}{5}$$

b) $I = \int_0^{\pi/2} x^2 \sin(2x)\cos(3x)\,dx$

Solution

$$I = \int_0^{\pi/2} x^2 \sin(2x)\cos(3x)\,dx = \int_0^{\pi/2} x^2 (1/2)\left[\sin(2x+3x) + \sin(2x-3x)\right] dx$$

$$= \frac{1}{2}\int_0^{\pi/2} x^2\left[\sin(5x) + \sin(-x)\right] dx =$$

$$= \frac{1}{2}\int_0^{\pi/2} x^2 \sin(5x)\,dx - \frac{1}{2}\int_0^{\pi/2} x^2 \sin x\,dx$$

Define $\forall a \in \mathbb{R}^+ : F(a) = \int_0^{\pi/2} x^2 \sin(ax)\,dx$

and note that

$$I = \frac{F(5) - F(1)}{2} \qquad (1)$$

↳ Note that it is more convenient to evaluate the general integral $F(a)$ and the use Eq. (1) than to do the two integrals separately.

$$F(a) = \int_0^{\pi/2} x^2 \sin(ax)\,dx = \int_0^{\pi/2} x^2 \left[\frac{-1}{a}\cos(ax)\right]' dx =$$

$$= \left[x^2 \cdot \frac{(-1)}{a}\cos(ax)\right]_0^{\pi/2} - \int_0^{\pi/2} (x^2)' \left[\frac{-1}{a}\cos(ax)\right] dx =$$

$$= -\left(\frac{\pi}{2}\right)^2 \frac{1}{a}\cos\left(\frac{a\pi}{2}\right) - \int_0^{\pi/2} 2x\left(\frac{-1}{a}\right)\cos(ax)\,dx =$$

$$= \frac{-\pi^2}{4a}\cos\left(\frac{a\pi}{2}\right) + \frac{2}{a}\int_0^{\pi/2} x\left[\frac{\sin(ax)}{a}\right]' dx =$$

$$= \frac{-\pi^2 \cos(a\pi/2)}{4a} + \frac{2}{a}\left[\frac{x\sin(ax)}{a}\right]_0^{\pi/2} - \frac{2}{a}\int_0^{\pi/2} \frac{(x)'\sin(ax)}{a}\,dx$$

$$= \frac{-\pi^2 \cos(a\pi/2)}{4a} + \frac{2}{a}\,\frac{\pi}{2}\,\frac{\sin(a\pi/2)}{a} - \frac{2}{a^2}\int_0^{\pi/2} \sin(ax)\,dx$$

$$= \frac{-\pi^2 \cos(a\pi/2)}{4a} + \frac{\pi \sin(a\pi/2)}{a^2} - \frac{2}{a^2}\left[\frac{-\cos(ax)}{a}\right]_0^{\pi/2}$$

$$= \frac{-\pi^2 \cos(a\pi/2)}{4a} + \frac{\pi \sin(a\pi/2)}{a^2} + \frac{2}{a^3}\left[\cos(a\pi/2) - 1\right]$$

and therefore

$$F(1) = \frac{-\pi^2 \cos(\pi/2)}{4} + \frac{\pi \sin(\pi/2)}{1^2} + \frac{2}{1^3}\left[\cos(\pi/2) - 1\right]$$

$$= 0 + \pi + 2[0-1] = \pi - 2$$

and

$$F(5) = \frac{-\pi^2 \cos(5\pi/2)}{4\cdot 5} + \frac{\pi \sin(5\pi/2)}{5^2} + \frac{2}{5^3}\left[\cos(5\pi/2) - 1\right]$$

$$= \frac{-\pi^2 \cos(2\pi + \pi/2)}{20} + \frac{\pi \sin(2\pi + \pi/2)}{25} + \frac{2}{125}\left[\cos(2\pi + \pi/2) - 1\right]$$

$$= \frac{-\pi^2 \cos(\pi/2)}{20} + \frac{\pi \sin(\pi/2)}{25} + \frac{2}{125}\left[\cos(\pi/2) - 1\right]$$

$$= 0 + \frac{\pi}{25} + \frac{2}{125}[0-1] = \frac{\pi}{25} - \frac{2}{125}$$

It follows that

$$I = (1/2)[F(5) - F(1)] = \frac{1}{2}\left[\left(\frac{\pi}{25} - \frac{2}{125}\right) - (\pi - 2)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{25} - \frac{2}{125} - \pi + 2\right] = \frac{1}{2}\,\frac{5\pi - 2 - 125\pi + 250}{125}$$

$$= \frac{1}{2}\,\frac{248-120\pi}{125} = \frac{1}{2}\,\frac{2(124-60\pi)}{125} =$$

$$= \frac{124-60\pi}{125}$$

↳ Note that more complicated products can be broken down in a similar manner one at a time.

For example:

$$\sin(5x)\cos(3x)\cos(7x) = \sin(5x)(1/2)[\cos(3x-7x)+\cos(3x+7x)]$$

$$= (1/2)\sin(5x)[\cos(-4x)+\cos(10x)] =$$

$$= (1/2)[\sin(5x)\cos(4x)+\sin(5x)\cos(10x)]$$

$$= (1/2)(1/2)[\sin(5x-4x)+\sin(5x+4x)+\sin(5x-10x)+\sin(5x+10x)]$$

$$= (1/4)[\sin x+\sin(9x)+\sin(-5x)+\sin(15x)]$$

$$= (1/4)(\sin x+\sin(9x)-\sin(5x)+\sin(15x))$$

EXERCISES

(31) Evaluate the following integrals:

a) $I = \int_0^{\pi/6} \sin(2x)\cos(3x)\,dx$ b) $I = \int_0^{\pi/3} \cos(5x)\cos(2x)\,dx$

c) $I = \int_0^{\pi/12} \sin(2x)\cdot\sin(4x)\,dx$ d) $I = \int_0^{3\pi/4} \sin x \cos(2x)\cos(3x)\,dx$

e) $I = \int_0^{\pi/6} \sin x \sin(2x)\sin(3x)\,dx$ f) $I = \int_0^{\pi/4} x^2\cos(3x)\sin(5x)\,dx$

g) $I = \int_0^{\pi/2} x^3\cos(2x)\cos(4x)\,dx$ h) $I = \int_0^{\pi} e^{-x}\sin x \sin(2x)\,dx$

i) $I = \int_0^{2\pi/3} e^{-x}\cos(2x)\sin(5x)\,dx$

j) $I = \int_0^{\pi/3} \sin\left(5x - \frac{\pi}{4}\right)\cos\left(x + \frac{\pi}{4}\right)dx$

k) $I = \int_0^{\pi} \sin\left(x - \frac{\pi}{3}\right)\cos\left(2x - \frac{\pi}{4}\right)dx$

(5) → **Form**: Squares of trigonometric functions

Squares of trigonometric functions can be eliminated by the half-angle formulas:

$\sin^2 x = \frac{1-\cos 2x}{2}$	$\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$
$\cos^2 x = \frac{1+\cos 2x}{2}$	$\cot^2 x = \frac{1+\cos 2x}{1-\cos 2x}$

EXAMPLES

a) $$I = \int_0^{\pi/2} \sin^2 x \cos^2 x \, dx = \frac{1}{4} \int_0^{\pi/2} (2\sin x \cos x)^2 \, dx =$$

$$= \frac{1}{4} \int_0^{\pi/2} \sin^2(2x) \, dx = \frac{1}{4} \int_0^{\pi/2} \frac{1-\cos 4x}{2} \, dx$$

$$= \frac{1}{8} \int_0^{\pi/2} dx - \frac{1}{8} \int_0^{\pi/2} \cos 4x \, dx =$$

$$= \frac{1}{8} \frac{\pi}{2} - \frac{1}{8} \left[\frac{\sin 4x}{4} \right]_0^{\pi/2} =$$

$$= \frac{1}{8} \left[\frac{\pi}{2} - \frac{\sin(2\pi) - \sin 0}{4} \right] =$$

$$= \frac{1}{8} \left[\frac{\pi}{2} - \frac{0-0}{4} \right] = \frac{\pi}{16}$$

b) $I = \int_0^{\pi/8} \tan^2 x \sin(2x)\,dx = \int_0^{\pi/8} \frac{1-\cos(2x)}{1+\cos(2x)} \sin(2x)\,dx$

Let $y = g(x) = \cos(2x) \Rightarrow \begin{cases} dy = -2\sin(2x)\,dx \\ g(0) = \cos(0) = 1 \\ g(\pi/8) = \cos(2\cdot(\pi/8)) = \cos(\pi/4) = \frac{\sqrt{2}}{2} \end{cases} \Rightarrow$

$\Rightarrow I = \int_1^{\sqrt{2}/2} \frac{1-y}{1+y} \cdot (-1/2)\,dy = -\frac{1}{2}\int_1^{\sqrt{2}/2} \frac{1-y}{1+y}\,dy$

$= -\frac{1}{2}\int_1^{\sqrt{2}/2} \frac{2-1-y}{1+y}\,dy = \frac{-1}{2}\int_1^{\sqrt{2}/2} \left[\frac{2}{1+y} - 1\right] dy =$

$= \frac{-1}{2}\left[2\ln|1+y| - y\right]_1^{\sqrt{2}/2} =$

$= \frac{-1}{2}\left[2\ln|1+\sqrt{2}/2| - \frac{\sqrt{2}}{2} - (2\ln|1+1| - 1)\right] =$

$= \frac{-1}{2}\left[2\ln(1+\sqrt{2}/2) - 2\ln 2 + 1 - \frac{\sqrt{2}}{2}\right]$

c) $I = \int_0^{\pi} \sqrt{1+\cos x}\ dx$

Solution

$$I = \int_0^{\pi} \sqrt{1+\cos x}\ dx = \int_0^{\pi} \sqrt{2\cos^2(x/2)}\ dx =$$

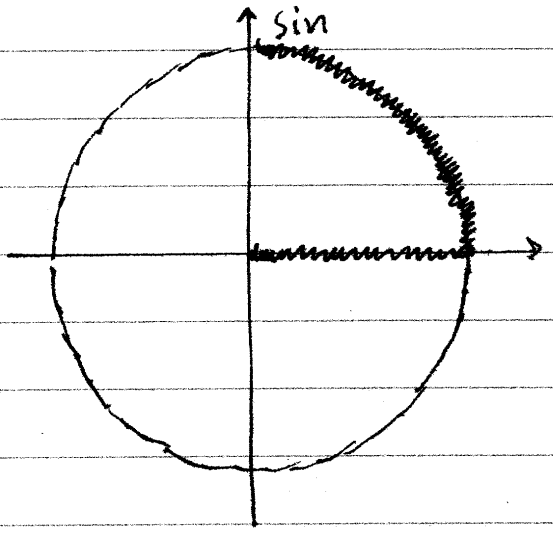
$$= \int_0^{\pi} \sqrt{2}\,|\cos(x/2)|\,dx$$

Let $x \in [0,\pi]$ be given. Then:

$x \in [0,\pi] \Rightarrow 0 \le x \le \pi \Rightarrow 0 \le x/2 \le \pi/2 \overset{(*)}{\Rightarrow} \cos(x/2) \ge 0 \Rightarrow$

$\Rightarrow |\cos(x/2)| = \cos(x/2)$

and therefore $\forall x \in [0,\pi]: |\cos(x/2)| = \cos(x/2)$



↳ We use the trig circle to justify the (*) step and note that:

$\sqrt{x^2} = |x|$

$(\sqrt{x})^2 = x$

It follows that

$$I = \int_0^{\pi} \sqrt{2}\cos(x/2)\,dx = \sqrt{2}\int_0^{\pi} \cos(x/2)\,dx = \sqrt{2}\left[\frac{\sin(x/2)}{1/2}\right]_0^{\pi} =$$

$$= 2\sqrt{2}\left[\sin(x/2)\right]_0^{\pi} = 2\sqrt{2}\left[\sin(\pi/2) - \sin 0\right] = 2\sqrt{2}(1-0)$$

$$= 2\sqrt{2}$$

EXERCISES

(32) Evaluate the following integrals.

a) $I = \int_0^{\pi/12} \cos^2 x \, dx$ b) $I = \int_0^{\pi/8} \sin^4 x \, dx$

c) $I = \int_0^{2\pi/3} \cos^6 x \, dx$ d) $I = \int_0^{\pi/4} \sin^2 x \tan^2 x \sin(2x) \, dx$

e) $I = \int_0^{\pi/8} \frac{\cos^2 x \sin(2x)}{\sin^2 x} \, dx$ f) $I = \int_0^{\pi/4} \sin^2 x \cos^4 x \, dx$

(33) Evaluate the following integrals.

a) $I = \int_0^{\pi/6} \sqrt{1-\cos x} \, dx$ b) $I = \int_0^{\pi/2} \sqrt{1+\sin(2x)} \, dx$

c) $I = \int_{\pi/6}^{\pi/2} \sqrt{1-\sin(3x)} \, dx$

↳ Note that for (b) and (c) we can use the cofunction identity:

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

to convert the signs into cosines. If necessary, to eliminate the absolute value, we may have to split the integral into subintervals

(34) Consider the integrals

$$I_1 = \int_0^{\pi/2} \sin^4 x \cos^2 x \, dx$$

$$I_2 = \int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$$

a) Evaluate $I_1 + I_2$ and $I_1 - I_2$.

b) Use the results of (a) to solve for I_1 and I_2.

↳ In the above argument use:

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x - \sin^2 x = \cos(2x)$

to simplify the calculation.

(35) Evaluate the following integrals

a) $I = \int_{\pi/4}^{\pi/2} \frac{dx}{\sin^4 x}$ b) $I = \int_{\pi/4}^{\pi/2} \frac{\cos^2 x \, dx}{\sin^4 x}$

c) $I = \int_{\pi/4}^{\pi/2} \frac{dx}{\sin^4 x \cos^2 x}$

↳ Introduce $\sin^2 x + \cos^2 x$ in the numerator to evaluate (a), using form $f(\tan x)/\cos^2 x$ or $f(\cot x)/\sin^2 x$. Then use (a) to do (B) and (c).

↳ In general integrals of the form $\int \sin^a x \cos^a x \, dx$ with a even & b even can be very challenging. Problems 32, 34, 35 illustrate some methods for handling such integrals.

⑥ → Form: $\boxed{I=\int f(\sin x, \cos x, \tan x, \cot x)\,dx}$

Method of desperation: We use this method if all trigonometric functions use the same angle x and we don't know what to do.

•1 Use the identities

$$\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)} \qquad \tan x = \frac{2\tan(x/2)}{1-\tan^2(x/2)}$$

$$\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} \qquad \cot x = \frac{1-\tan^2(x/2)}{2\tan(x/2)}$$

to rewrite the integral as

$$\boxed{I=\int F(\tan(x/2))\,dx}$$

•2 Let $y=\tan(x/2)$. It follows that

$$dy = (1/2)(1+\tan^2(x/2))\,dx = (1/2)(1+y^2)\,dx \rightarrow$$

$$\Rightarrow dx = \frac{2}{1+y^2}\,dy \Rightarrow \boxed{I=\int F(y)\frac{2}{1+y^2}\,dy} = \dots$$

thus the integral reduces to a rational integral.

↕ → The above method is known as tangent substitution.

EXAMPLES

a) $I = \int \frac{d\vartheta}{3-\cos\vartheta}$

Let $y = \tan(\vartheta/2) \Rightarrow dy = (1/2)(1+\tan^2(\vartheta/2))\,d\vartheta =$

$= (1/2)(1+y^2)\,d\vartheta \Rightarrow$

$\Rightarrow d\vartheta = \frac{2\,dy}{1+y^2}$ ∎

Note that $\cos\vartheta = \frac{1-\tan^2(\vartheta/2)}{1+\tan^2(\vartheta/2)} = \frac{1-y^2}{1+y^2}$. It follows that

$$I = \int \frac{\frac{2}{1+y^2}}{3-\frac{1-y^2}{1+y^2}}\,dy = \int \frac{2}{3(1+y^2)-(1-y^2)}\,dy =$$

$$= \int \frac{2}{3+3y^2-1+y^2}\,dy = \int \frac{2}{4y^2+2}\,dy =$$

$$= \int \frac{dy}{2y^2+1}. \quad \text{Let } z = \sqrt{2}\,y \Rightarrow dz = \sqrt{2}\,dy \Rightarrow$$

$$\Rightarrow dy = (1/\sqrt{2})\,dz \Rightarrow$$

$$\Rightarrow I = \int \frac{(1/\sqrt{2})\,dz}{z^2+1} = \frac{1}{\sqrt{2}}\operatorname{Arctan}(z) + c =$$

$$= \frac{1}{\sqrt{2}}\operatorname{Arctan}(\sqrt{2}\,y) + c =$$

$$= \frac{1}{\sqrt{2}}\operatorname{Arctan}(\sqrt{2}\tan(\vartheta/2)) + c$$

B) Evaluate the following integral

$$I=\int_0^{\pi/4}\frac{dx}{(3\sin x+\cos x)^2}$$

Solution

We note that

$$\begin{aligned}(3\sin x+\cos x)^2 &= 9\sin^2 x+6\sin x\cos x+\cos^2 x=\\ &=(\sin^2 x+\cos^2 x)+8\sin^2 x+3(2\sin x\cos x)\\ &=1+4(2\sin^2 x)+3\sin(2x)\\ &=1+4(1-\cos(2x))+3\sin(2x)\\ &=1+4-4\cos(2x)+3\sin(2x)\\ &=5-4\cos(2x)+3\sin(2x)\equiv f(x)\end{aligned}$$

and therefore

$$I=\int_0^{\pi/4}\frac{dx}{5-4\cos(2x)+3\sin(2x)}$$

Let $y=\tan x$. It follows that

$$dy=(\tan x)'\,dx=(1+\tan^2 x)\,dx=(1+y^2)\,dx \Rightarrow$$

$$\Rightarrow dx=\frac{dy}{1+y^2}$$

and

$$\cos(2x)=\frac{1-\tan^2 x}{1+\tan^2 x}=\frac{1-y^2}{1+y^2}$$

$$\sin(2x)=\frac{2\tan x}{1+\tan^2 x}=\frac{2y}{1+y^2}$$

therefore,

$$f(x) = 5 - 4\cos(2x) + 3\sin(2x)$$

$$= 5 - 4\left(\frac{1-y^2}{1+y^2}\right) + 3\left(\frac{2y}{1+y^2}\right)$$

$$= \frac{5(1+y^2) - 4(1-y^2) + 3(2y)}{1+y^2}$$

$$= \frac{5+5y^2-4+4y^2+6y}{1+y^2} = \frac{(5+4)y^2+6y+(5-4)}{1+y^2}$$

$$= \frac{9y^2+6y+1}{y^2+1} = \frac{(3y+1)^2}{y^2+1}$$

For $x = 0 \Rightarrow y = \tan 0 = 0$

For $x = \pi/4 \Rightarrow y = \tan(\pi/4) = 1$

It follows that

$$I = \int_0^1 \frac{y^2+1}{(3y+1)^2}\,\frac{dy}{y^2+1} = \int_0^1 \frac{dy}{(3y+1)^2} =$$

$$= \frac{1}{9}\int_0^1 \frac{dy}{(y+1/3)^2} = \frac{1}{9}\int_{1/3}^{4/3} \frac{dy}{y^2} =$$

$$= \frac{1}{9}\left[\frac{y^{-1}}{-1}\right]_{1/3}^{4/3} = \frac{-1}{9}\left[\frac{1}{y}\right]_{1/3}^{4/3} =$$

$$= \frac{-1}{9}\left[\frac{1}{4/3} - \frac{1}{1/3}\right] = \frac{-1}{9}\left[\frac{3}{4} - 3\right]$$

$$= \frac{-3}{9}\left[\frac{1}{4} - 1\right] = \frac{-1}{3}\left(\frac{-3}{4}\right) = \frac{1}{4}$$

EXERCISES

(36) Evaluate the following integrals using tangent substitution

a) $I = \int_0^{\pi/4} \frac{dx}{\sin x + \cos x}$

b) $I = \int_0^{\pi/3} \frac{dx}{\cos x}$

c) $I = \int_0^{\pi/6} \frac{dx}{1+\sin x}$

d) $I = \int_0^{\pi/2} \frac{dx}{1+\sin x+\cos x}$

e) $I = \int_0^{\pi/6} \frac{dx}{\sin x + 3\cos x}$

f) $I = \int_0^{\pi/4} \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$

g) $I = \int \frac{dx}{2\cos x - 3\tan x}$

(37) Evaluate the following integrals using tangent substitution.

a) $I = \int_0^{\pi/4} \frac{dx}{(\sin x + \cos x)^2}$

b) $I = \int \frac{dx}{\sin^2 x - 4\cos^2 x}$

c) $I = \int_0^{\pi/6} \frac{dx}{5 + 2\cos^2 x}$

↳ The following identities can be used to simplify the integrand and to reduce powers before applying the tangent substitution.

$2\sin^2 x = 1 - \cos(2x)$ $\qquad \sin^2 x + \cos^2 x = 1$

$2\cos^2 x = 1 + \cos(2x)$

$2\sin x \cos x = \sin(2x)$

Rationalizing substitutions

Integrals that follow the general form

$$I=\int F(x,\sqrt{ax^2+bx+c})\,dx$$

can be evaluated either by trigonometric substitution or an algebraic substitution (or both), as follows:

(I) → $\boxed{I=\int F(x,\sqrt{a^2-(bx+c)^2})\,dx}$

For integrals where the coefficient of x^2 in the radical is negative, the only method available is to complete squares and let

$\boxed{bx+c=a\sin\vartheta \quad \text{with } \vartheta\in[-\pi/2,\pi/2]}$

It follows that

$bx+c=a\sin\vartheta \Leftrightarrow bx=a\sin\vartheta-c \Leftrightarrow x=(1/b)(a\sin\vartheta-c)$

and therefore

$dx=(1/b)(a\sin\vartheta-c)'\,d\vartheta=(a/b)\cos\vartheta\,d\vartheta$

and

$a^2-(bx+c)^2=a^2-a^2\sin^2\vartheta=a^2(1-\sin^2\vartheta)=a^2\cos^2\vartheta$

$\Rightarrow\sqrt{a^2-(bx+c)^2}=\sqrt{a^2\cos^2\vartheta}=|a||\cos\vartheta|=|a|\cos\vartheta$

because $\vartheta\in[-\pi/2,\pi/2]\Rightarrow\cos\vartheta\geq 0$. We conclude that

$$I=\int F((1/b)(a\sin\vartheta-c),|a|\cos\vartheta)(a/b)\cos\vartheta\,d\vartheta$$

reducing the integral to a trigonometric integral.

EXAMPLES

a) Evaluate $I = \int_{-1}^{1} \frac{x^2+x+1}{\sqrt{4-x^2}}\, dx$

Solution

Let $x = 2\sin\vartheta$ with $\vartheta \in [-\pi/2, \pi/2]$. Then:

$dx = (2\sin\vartheta)'\, d\vartheta = 2\cos\vartheta\, d\vartheta$

and

$4-x^2 = 4-(2\sin\vartheta)^2 = 4-4\sin^2\vartheta = 4(1-\sin^2\vartheta) = 4\cos^2\vartheta \Rightarrow$

$\Rightarrow \sqrt{4-x^2} = \sqrt{4\cos^2\vartheta} = 2|\cos\vartheta| = 2\cos\vartheta$

For $x=-1 \Leftrightarrow 2\sin\vartheta = -1 \Leftrightarrow \sin\vartheta = -1/2 = -\sin(\pi/6) = \sin(-\pi/6)$

$\Leftrightarrow \vartheta = -\pi/6$

For $x=1 \Leftrightarrow 2\sin\vartheta = 1 \Leftrightarrow \sin\vartheta = 1/2 = \sin(\pi/6) \Leftrightarrow$

$\Leftrightarrow \vartheta = \pi/6$

It follows that:

$$I = \int_{-\pi/6}^{\pi/6} \frac{(2\sin\vartheta)^2 + 2\sin\vartheta + 1}{2\cos\vartheta}\, 2\cos\vartheta\, d\vartheta =$$

$$= \int_{-\pi/6}^{\pi/6} [4\sin^2\vartheta + 2\sin\vartheta + 1]\, d\vartheta =$$

$$= \int_{-\pi/6}^{\pi/6} [2(1-\cos(2\vartheta)) + 2\sin\vartheta + 1]\, d\vartheta =$$

$$= \int_{-\pi/6}^{\pi/6} (3 - 2\cos(2\vartheta) + 2\sin\vartheta)\, d\vartheta$$

$$= \left[3\vartheta - \frac{2\sin(2\vartheta)}{2} - 2\cos\vartheta \right]_{-\pi/6}^{\pi/6}$$

$$= [3\vartheta - \sin(2\vartheta) - 2\cos\vartheta]_{-\pi/6}^{\pi/6} =$$

$$= [3(\pi/6) - \sin(\pi/3) - 2\cos(\pi/6)] -$$

$$-[3(-\pi/6) - \sin(-\pi/3) - 2\cos(-\pi/6)] =$$

$$= \pi/2 - \sqrt{3}/2 - \underline{2\sqrt{3}/2} + \pi/2 - \sqrt{3}/2 + \underline{2\sqrt{3}/2}$$

$$= \pi - \sqrt{3}.$$

B) Evaluate $I = \int_0^2 \frac{x}{\sqrt{4+4x-x^2}}\, dx$

Solution

We note that

$4+4x-x^2 = 4-(x^2-4x) = 4-(x^2-4x+4)+4 =$

$= 8-(x-2)^2$

and therefore let $x-2 = \sqrt{8}\sin\vartheta$ with $\vartheta \in [-\pi/2, \pi/2]$.

Then: $x = 2+\sqrt{8}\sin\vartheta$

and $dx = (2+\sqrt{8}\sin\vartheta)'\, d\vartheta = \sqrt{8}\cos\vartheta\, d\vartheta$

and

$4+4x-x^2 = 8-(x-2)^2 = 8-(\sqrt{8}\sin\vartheta)^2 =$

$= 8-8\sin^2\vartheta = 8(1-\sin^2\vartheta) = 8\cos^2\vartheta \Rightarrow$

$\Rightarrow \sqrt{4+4x-x^2} = \sqrt{8\cos^2\vartheta} = \sqrt{8}\,|\cos\vartheta| = \sqrt{8}\cos\vartheta$

For $x=0 \Leftrightarrow 2+\sqrt{8}\sin\vartheta = 0 \Leftrightarrow 2+2\sqrt{2}\sin\vartheta = 0$

$\Leftrightarrow 2\sqrt{2}\sin\vartheta = -2 \Leftrightarrow$

$\Leftrightarrow \sin\vartheta = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2} = -\sin(\pi/4)$

$= \sin(-\pi/4) \Leftrightarrow \vartheta = -\pi/4$

For $x=2 \Leftrightarrow 2+\sqrt{8}\sin\vartheta = 2 \Leftrightarrow \sqrt{8}\sin\vartheta = 0 \Leftrightarrow \sin\vartheta = 0$

$\Leftrightarrow \vartheta = 0$

It follows that

$I = \int_{-\pi/4}^{0} \frac{2+\sqrt{8}\sin\vartheta}{\sqrt{8}\cos\vartheta}\sqrt{8}\cos\vartheta\, d\vartheta =$

$= \int_{-\pi/4}^{0} (2+2\sqrt{2}\sin\vartheta)\, d\vartheta$

$$= \left[2\vartheta + 2\sqrt{2}(-\cos\vartheta)\right]_{-\pi/4}^{0} =$$

$$= \left[2\cdot 0 - 2\sqrt{2}\cos 0\right] - \left[2(-\pi/4) - 2\sqrt{2}\cos(-\pi/4)\right]$$

$$= -2\sqrt{2} + \pi/2 + 2\sqrt{2}\cos(\pi/4)$$

$$= -2\sqrt{2} + \pi/2 + 2\sqrt{2}(\sqrt{2}/2)$$

$$= -2\sqrt{2} + \pi/2 + 2$$

EXERCISE

(38) Evaluate the following integrals

a) $I = \int_0^{\sqrt{2}} \sqrt{2-x^2}\, dx$

b) $I = \int \frac{x^3 dx}{\sqrt{1-x^2}}$

c) $I = \int \frac{(x-1)(x+3)}{\sqrt{9-2x^2}}\, dx$

d) $I = \int_0^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$

e) $I = \int_0^{1/2} \frac{x^2 dx}{1+\sqrt{1-x^2}}$

f) $I = \int \frac{dx}{x+3\sqrt{9-x^2}}$

g) $I = \int \sqrt{1+x-x^2}\, dx$

h) $I = \int x^2\sqrt{9+6x-x^2}\, dx$

i) $I = \int \frac{dx}{\sqrt{6+5x-x^2}}$

j) $I = \int \frac{x(x+3)\, dx}{\sqrt{25+10x-x^2}}$

(2) → $\boxed{I = \int F\left(x, \sqrt{ax^2+bx+c}\right) dx \quad \text{with } a>0}$

Algebraic rationalizing substitution:

Let $\boxed{y - x\sqrt{a} = \sqrt{ax^2+bx+c}}$ (1)

• 1 We solve for x:

Eq.(1) $\Leftrightarrow (y - x\sqrt{a})^2 = \left(\sqrt{ax^2+bx+c}\right)^2 \Leftrightarrow$

$\Leftrightarrow y^2 - 2xy\sqrt{a} + ax^2 = ax^2 + bx + c \Leftrightarrow$

$\Leftrightarrow y^2 - 2xy\sqrt{a} = bx + c \Leftrightarrow (2y\sqrt{a}+b)x = y^2 - c$

$\Leftrightarrow x = \dfrac{y^2-c}{2y\sqrt{a}+b}$

• 2 We calculate dx in terms of y:

$$dx = \left[\frac{y^2-c}{2y\sqrt{a}+b}\right]' dy =$$

$$= \frac{(y^2-c)'(2y\sqrt{a}+b) - (y^2-c)(2y\sqrt{a}+b)'}{(2y\sqrt{a}+b)^2} dy$$

$$= \frac{2y(2y\sqrt{a}+b) - (y^2-c)2\sqrt{a}}{(2y\sqrt{a}+b)^2} dy$$

$$= \frac{4y^2\sqrt{a} + 2by - 2y^2\sqrt{a} + 2c\sqrt{a}}{(2y\sqrt{a}+b)^2} dy$$

$$= \frac{2y^2\sqrt{a} + 2by + 2c\sqrt{a}}{(2y\sqrt{a}+b)^2} dy$$

$$= \frac{2(y^2\sqrt{a} + by + c\sqrt{a})}{(2y\sqrt{a}+b)^2} dy$$

• 3 We calculate $\sqrt{ax^2+bx+c}$ in terms of y

$$\sqrt{ax^2+bx+c} = y - x\sqrt{a} = y - \frac{y^2-c}{2y\sqrt{a}+b}\sqrt{a} =$$

$$= \frac{y(2y\sqrt{a}+b) - (y^2-c)\sqrt{a}}{2y\sqrt{a}+b} =$$

$$= \frac{2y^2\sqrt{a}+by-y^2\sqrt{a}+c\sqrt{a}}{2y\sqrt{a}+b} =$$

$$= \frac{y^2\sqrt{a}+by+c\sqrt{a}}{2y\sqrt{a}+b}$$

• 4 It follows that:

$$I = \int F\left(\frac{y^2-c}{2y\sqrt{a}+b}, \frac{y^2\sqrt{a}+by+c\sqrt{a}}{2y\sqrt{a}+b}\right) \frac{2(y^2\sqrt{a}+by+c\sqrt{a})}{(2y\sqrt{a}+b)^2}\,dy$$

EXAMPLES

a) $I = \int_0^2 \frac{dx}{\sqrt{x^2+3x+5}}$

Solution

Let $y - x = \sqrt{x^2+3x+5} \Leftrightarrow (y-x)^2 = x^2+3x+5 \Leftrightarrow$
$\Leftrightarrow y^2 - 2xy + x^2 = x^2+3x+5 \Leftrightarrow y^2 - 2xy = 3x+5$
$\Leftrightarrow 3x + 2xy = y^2 - 5 \Leftrightarrow x(3+2y) = y^2 - 5 \Leftrightarrow x = \frac{y^2-5}{2y+3}$

Then:

$$dx = \left(\frac{y^2-5}{2y+3}\right)' dy = \frac{(y^2-5)'(2y+3) - (y^2-5)(2y+3)'}{(2y+3)^2} dy =$$

$$= \frac{2y(2y+3) - 2(y^2-5)}{(2y+3)^2} dy = \frac{4y^2+6y-2y^2+10}{(2y+3)^2} dy$$

$$= \frac{2y^2+6y+10}{(2y+3)^2} dy = \frac{2(y^2+3y+5)}{(2y+3)^2} dy$$

and

$$\sqrt{x^2+3x+5} = y - x = y - \frac{y^2-5}{2y+3} = \frac{y(2y+3) - (y^2-5)}{2y+3}$$

$$= \frac{2y^2+3y-y^2+5}{2y+3} = \frac{y^2+3y+5}{2y+3}$$

Since $y = x + \sqrt{x^2+3x+5}$,

$x = 0 \Rightarrow y = 0 + \sqrt{0+0+5} = \sqrt{5}$

$x = 2 \Rightarrow y = 2 + \sqrt{2^2+3\cdot 2+5} = 2 + \sqrt{4+6+5} = 2+\sqrt{15}$

From the above, it follows that:

$$I = \int_{\sqrt{5}}^{2+\sqrt{15}} \frac{\left[\dfrac{2(y^2+3y+5)}{(2y+3)^2}\right]}{\left[\dfrac{y^2+3y+5}{2y+3}\right]}\,dy - \int_{\sqrt{5}}^{2+\sqrt{15}} \frac{2\,dy}{2y+3}$$

$$= \int_{\sqrt{5}}^{2+\sqrt{15}} \frac{(2y+3)'\,dy}{2y+3} = \Big[\ln|2y+3|\Big]_{\sqrt{5}}^{2+\sqrt{15}} =$$

$$= \ln|2(2+\sqrt{15})+3| - \ln|2\sqrt{5}+3|$$

$$= \ln|4+2\sqrt{15}+3| - \ln|3+2\sqrt{5}|$$

$$= \ln(7+2\sqrt{15}) - \ln(3+2\sqrt{5})$$

b) $I = \int_2^4 \frac{\sqrt{x^2-4}}{x}\, dx$

Solution

Let $y - x = \sqrt{x^2-4} \Leftrightarrow (y-x)^2 = x^2-4 \Leftrightarrow y^2 - 2xy + x^2 = x^2 - 4$
$\Leftrightarrow y^2 - 2xy = -4 \Leftrightarrow 2xy = y^2+4 \Leftrightarrow x = \frac{y^2+4}{2y}$

We note that

$$dx = \left[\frac{y^2+4}{2y}\right]' dy = \frac{(y^2+4)'(2y) - (y^2+4)(2y)'}{(2y)^2} dy =$$

$$= \frac{(2y)(2y) - 2(y^2+4)}{4y^2} dy = \frac{4y^2 - 2y^2 - 8}{4y^2} dy =$$

$$= \frac{2y^2-8}{4y^2} dy = \frac{y^2-4}{2y^2} dy$$

and

$$\sqrt{x^2-4} = y - x = y - \frac{y^2+4}{2y} = \frac{2y^2 - (y^2+4)}{2y} =$$

$$= \frac{2y^2 - y^2 - 4}{2y} = \frac{y^2-4}{2y}$$

and since $y = x + \sqrt{x^2-4}$

$x = 2 \Rightarrow y = 2 + \sqrt{2^2-4} = 2 + \sqrt{4-4} = 2$

$x = 4 \Rightarrow y = 4 + \sqrt{4^2-4} = 4 + \sqrt{16-4} = 4 + \sqrt{12}$

$= 4 + 2\sqrt{3}$

From the above, it follows that:

$$I = \int_2^{4+2\sqrt{3}} \frac{\left[\frac{y^2-4}{2y}\right]}{\left[\frac{y^2+4}{2y}\right]} \frac{y^2-4}{2y^2}\,dy$$

$$= \int_2^{4+2\sqrt{3}} \frac{(y^2-4)^2}{2y^2(y^2+4)}\,dy$$

It is sufficient to do the partial fraction decomposition of

$$g(z) = \frac{(z-4)^2}{2z(z+4)} = \frac{z^2-8z+16}{2z^2+8z}$$

Divide $z^2-8z+16$ with $2z^2+8z$

$z^2-8z+16$	$2z^2+8z$
$-z^2-4z$	$(1/2)$
$-12z+16$	

thus:

$$g(z) = \frac{1}{2} + \frac{-12z+16}{2z(z+4)} = \frac{1}{2} + \frac{-6z+8}{z(z+4)} =$$

$$= \frac{1}{2} + \frac{A}{z} + \frac{B}{z+4}$$

with

$$A = \left.\frac{-6z+8}{z+4}\right|_{z=0} = \frac{-0+8}{0+4} = 2$$

$$B = \left.\frac{-6z+8}{z}\right|_{z=-4} = \frac{-6(-4)+8}{-4} = \frac{24+8}{-4} =$$

$$= -6-2 = -8$$

and therefore,

$$\frac{(z-4)^2}{2z(z+4)} = \frac{1}{2} + \frac{2}{z} - \frac{8}{z+4} \Rightarrow$$

$$\Rightarrow \frac{(y^2-4)^2}{2y^2(y^2+4)} = \frac{1}{2} + \frac{2}{y^2} - \frac{8}{y^2+4} \Rightarrow$$

$$\Rightarrow I = \int_2^{4+2\sqrt{3}} \frac{dy}{2} + \int_2^{4+2\sqrt{3}} \frac{2dy}{y^2} - \int_2^{4+2\sqrt{3}} \frac{8dy}{y^2+4}$$

$$= I_1 + I_2 + I_3$$

Since,

$$I_1 = \int_2^{4+2\sqrt{3}} \frac{dy}{2} = \frac{1}{2}\left[y\right]_2^{4+2\sqrt{3}} = \frac{(4+2\sqrt{3})-2}{2}$$

$$= \frac{2+2\sqrt{3}}{2} = 1+\sqrt{3}$$

and

$$I_2 = \int_2^{4+2\sqrt{3}} \frac{2dy}{y^2} = \left[\frac{2y^{-1}}{-1}\right]_2^{4+2\sqrt{3}} = \left[\frac{-2}{y}\right]_2^{4+2\sqrt{3}} =$$

$$= \frac{-2}{4+2\sqrt{3}} - \frac{-2}{2} = 1 - \frac{1}{2+\sqrt{3}} =$$

$$= 1 - \frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = 1 - \frac{2-\sqrt{3}}{4-3} =$$

$$= 1 - (2-\sqrt{3}) = 1 - 2 + \sqrt{3} = \sqrt{3} - 1$$

and

$$I_3 = \int_2^{4+2\sqrt{3}} \frac{-8dy}{y^2+4} = -8\int_2^{4+2\sqrt{3}} \frac{dy}{y^2+2^2} =$$

$$= -8\left[\frac{1}{2}\operatorname{Arctan}\left(\frac{y}{2}\right)\right]_2^{4+2\sqrt{3}} =$$

$$= -8\left[\frac{1}{2}\operatorname{Arctan}\left(\frac{4+2\sqrt{3}}{2}\right) - \frac{1}{2}\operatorname{Arctan}\left(\frac{2}{2}\right)\right]$$

$$= -4\left[\operatorname{Arctan}(2+\sqrt{3}) - \operatorname{Arctan}(1)\right]$$

$$= -4\left[\operatorname{Arctan}(2+\sqrt{3}) - \pi/4\right] =$$

$$= \pi - 4\operatorname{Arctan}(2+\sqrt{3})$$

It follows that

$$I = I_1 + I_2 + I_3 =$$

$$= (1+\sqrt{3}) + (\sqrt{3}-1) + (\pi - 4\operatorname{Arctan}(2+\sqrt{3}))$$

$$= 2\sqrt{3} + \pi - 4\operatorname{Arctan}(2+\sqrt{3}).$$

EXERCISE

(39) Evaluate the following integrals

a) $I = \int \frac{dx}{\sqrt{x^2+3x+2}}$

b) $I = \int_0^1 \frac{dx}{\sqrt{x^2+5x+7}}$

c) $I = \int \frac{x^2 dx}{\sqrt{x^2+x+1}}$

d) $I = \int_0^1 \frac{x^3 dx}{\sqrt{x^2+x+3}}$

(3) $\longrightarrow$ $\boxed{I = \int F\left(x, \sqrt{(ax+b)^2 - c^2}\right) dx}$

Trigonometric substitution:

Let $\boxed{ax+b = \dfrac{c}{\cos\vartheta} \quad \text{with} \quad \vartheta \in [0, \pi/2) \cup (\pi/2, \pi]}$

• 1 Solve for x:

$$ax+b = \frac{c}{\cos\vartheta} \Leftrightarrow ax = \frac{c}{\cos\vartheta} - b = \frac{c - b\cos\vartheta}{\cos\vartheta} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{c - b\cos\vartheta}{a\cos\vartheta}$$

• 2 Calculate dx

$$dx = \left[\frac{c - b\cos\vartheta}{a\cos\vartheta}\right]' d\vartheta = \left[\frac{c}{a\cos\vartheta} - \frac{b}{a}\right]' d\vartheta =$$

$$= \frac{-c(\cos\vartheta)'}{a\cos^2\vartheta} d\vartheta = \frac{c\sin\vartheta}{a\cos^2\vartheta} d\vartheta$$

• 3 Calculate $\sqrt{(ax+b)^2 - c^2}$

$$(ax+b)^2 - c^2 = \frac{c^2}{\cos^2\vartheta} - c^2 = \frac{c^2 - c^2\cos^2\vartheta}{\cos^2\vartheta} = \frac{c^2(1-\cos^2\vartheta)}{\cos^2\vartheta} =$$

$$= \frac{c^2\sin^2\vartheta}{\cos^2\vartheta} = c^2\tan^2\vartheta \Rightarrow$$

$$\Rightarrow \sqrt{(ax+b)^2 - c^2} = \sqrt{c^2\tan^2\vartheta} = |c||\tan\vartheta| =$$

$$= \begin{cases} |c|\tan\vartheta & , \text{if } \vartheta \in [0, \pi/2) \\ -|c|\tan\vartheta & , \text{if } \vartheta \in (\pi/2, \pi] \end{cases}$$

• 4 It follows that

$$I = \int F\left(\frac{c - b\cos\vartheta}{\cos\vartheta}, |c||\tan\vartheta|\right) \frac{c\sin\vartheta}{\cos^2\vartheta} d\vartheta$$

which is a trigonometric integral reducible to a rational integral.

EXAMPLE

a) $I=\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$

Solution

Let $x=\frac{2}{\cos\vartheta}$ with $\vartheta\in[0,\pi/2)\cup(\pi/2,\pi]$. Then:

$$dx=\left(\frac{2}{\cos\vartheta}\right)'d\vartheta=\frac{-2(\cos\vartheta)'}{\cos^2\vartheta}d\vartheta=\frac{-2(-\sin\vartheta)}{\cos^2\vartheta}d\vartheta$$

$$=\frac{2\sin\vartheta}{\cos^2\vartheta}d\vartheta=\frac{2\tan\vartheta}{\cos\vartheta}d\vartheta$$

and

$$x^2-4=\left(\frac{2}{\cos\vartheta}\right)^2-4=\frac{4}{\cos^2\vartheta}-4=\frac{4(1-\cos^2\vartheta)}{\cos^2\vartheta}=$$

$$=\frac{4\sin^2\vartheta}{\cos^2\vartheta}=4\tan^2\vartheta\Rightarrow\sqrt{x^2-4}=2|\tan\vartheta|$$

For $x=2\Leftrightarrow\frac{2}{\cos\vartheta}=2\Leftrightarrow\cos\vartheta=1\Leftrightarrow\vartheta=0$

$x=4\Leftrightarrow\frac{2}{\cos\vartheta}=4\Leftrightarrow\cos\vartheta=\frac{1}{2}=\cos\left(\frac{\pi}{3}\right)\Leftrightarrow\vartheta=\frac{\pi}{3}$

It follows that:

$$I=\int_0^{\pi/3}\frac{2|\tan\vartheta|}{\left(\frac{2}{\cos\vartheta}\right)}\frac{2\tan\vartheta}{\cos\vartheta}d\vartheta=$$

$$= \int_0^{\pi/3} 2|\tan\vartheta|\tan\vartheta\, d\vartheta = \int_0^{\pi/3} 2\tan^2\vartheta\, d\vartheta$$

Let $y = \tan\vartheta \Rightarrow dy = (\tan\vartheta)' d\vartheta = (1+\tan^2\vartheta)\, d\vartheta = (1+y^2)\, d\vartheta$

$$\Rightarrow d\vartheta = \frac{dy}{1+y^2}$$

For $\vartheta = 0 \Rightarrow y = \tan 0 = 0$

For $\vartheta = \pi/3 \Rightarrow y = \tan(\pi/3) = \sqrt{3}$

and it follows that

$$I = \int_0^{\sqrt{3}} 2y^2 \frac{dy}{1+y^2} = 2\int_0^{\sqrt{3}} \frac{y^2}{1+y^2}\, dy = 2\int_0^{\sqrt{3}} \frac{(1+y^2)-1}{1+y^2}\, dy$$

$$= 2\int_0^{\sqrt{3}} \left[1 - \frac{1}{1+y^2}\right] dy = 2\Big[y - \text{Arctan}(y)\Big]_0^{\sqrt{3}}$$

$$= 2\left[\sqrt{3} - \text{Arctan}(\sqrt{3})\right] - 2\left[0 - \text{Arctan}(0)\right]$$

$$= 2\left[\sqrt{3} - \pi/3\right] - 2(0-0)$$

$$= 2\sqrt{3} - \frac{2\pi}{3}$$

↳ We have calculated this integral using both the algebraic and trigonometric methods and found

$I = 2\sqrt{3} + \pi - 4\,\text{Arctan}(2+\sqrt{3})$

$I = 2\sqrt{3} - 2\pi/3$

These are in fact equal to each other and it follows that:

$$\pi - 4\,\text{Arctan}(2+\sqrt{3}) = -2\pi/3 \Leftrightarrow 4\,\text{Arctan}(2+\sqrt{3}) = \pi + 2\pi/3$$

$$\Leftrightarrow 4\,\text{Arctan}(2+\sqrt{3}) = 5\pi/3 \Leftrightarrow \text{Arctan}(2+\sqrt{3}) = 5\pi/12.$$

EXERCISE

(40) Evaluate the following integrals

a) $I = \int \sqrt{x^2-5}\, dx$

b) $I = \int x^2\sqrt{x^2-1}\, dx$

c) $I = \int_0^{1/2} \frac{x^3\, dx}{\sqrt{x^2-1}}$

d) $I = \int_0^{-2} \sqrt{x^2+6x+8}\, dx$

(4) → $\boxed{I = \int F(x, \sqrt{(ax+b)^2 + c^2})\, dx}$

Trigonometric substitution

Let $\boxed{ax+b = c\tan\vartheta \quad \text{with} \quad \vartheta \in (-\pi/2, \pi/2)}$

• 1 Solve for x:

$ax+b = c\tan\vartheta \Leftrightarrow ax = c\tan\vartheta - b \Leftrightarrow x = (1/a)(c\tan\vartheta - b)$

• 2 Calculate dx:

$dx = (1/a)(c\tan\vartheta - b)'\, d\vartheta = \dfrac{c}{a\cos^2\vartheta}\, d\vartheta$

• 3 Calculate $\sqrt{(ax+b)^2 + c^2}$

$(ax+b)^2 + c^2 = [c\tan\vartheta]^2 + c^2 = c^2(1+\tan^2\vartheta) = \dfrac{c^2}{\cos^2\vartheta} \Rightarrow$

$$\Rightarrow \sqrt{(ax+b)^2 + c^2} = \sqrt{\frac{c^2}{\cos^2\vartheta}} = \left|\frac{c}{\cos\vartheta}\right| = \frac{|c|}{|\cos\vartheta|}$$

$$= \frac{|c|}{\cos\vartheta}$$

noting that $\vartheta \in (-\pi/2, \pi/2) \Rightarrow \cos\vartheta > 0$.

• 4 It follows that:

$$I = \int F\left(\frac{c\tan\vartheta - b}{a}, \frac{|c|}{\cos\vartheta}\right) \frac{c}{a\cos^2\vartheta}\, d\vartheta$$

which is a trigonometric integral that can be reduced to a rational integral.

EXAMPLE

Evaluate $I=\int_{-3}^{-2} x\sqrt{x^2+4x+5}\,dx$

Solution

We note that

$x^2+4x+5=(x^2+4x+4)+1=(x+2)^2+1$

so we define $x+2=\tan\vartheta$ with $\vartheta\in(-\pi/2,\pi/2)$.

It follows that

$x=\tan\vartheta-2$

$dx=(\tan\vartheta-2)'d\vartheta=\dfrac{d\vartheta}{\cos^2\vartheta}$

and

$x^2+4x+5=(x+2)^2+1=\tan^2\vartheta+1=\dfrac{1}{\cos^2\vartheta}\Rightarrow$

$\Rightarrow\sqrt{x^2+4x+5}=\sqrt{\dfrac{1}{\cos^2\vartheta}}=\dfrac{1}{|\cos\vartheta|}=\dfrac{1}{\cos\vartheta}$

because $\vartheta\in(-\pi/2,\pi/2)\Rightarrow\cos\vartheta>0\Rightarrow|\cos\vartheta|=\cos\vartheta$

For $x=-3\Leftrightarrow\tan\vartheta-2=-3\Leftrightarrow\tan\vartheta=2-3=-1$

$\Leftrightarrow\vartheta=\operatorname{Arctan}(-1)=-\operatorname{Arctan}(1)=-\pi/4$

For $x=-2\Leftrightarrow\tan\vartheta-2=-2\Leftrightarrow\tan\vartheta=0\Leftrightarrow\vartheta=0$

It follows that

$I=\int_{-\pi/4}^{0}(\tan\vartheta-2)\dfrac{1}{\cos\vartheta}\dfrac{d\vartheta}{\cos^2\vartheta}=$

$=\int_{-\pi/4}^{0}\left[\dfrac{\sin\vartheta}{\cos\vartheta}-2\right]\dfrac{d\vartheta}{\cos^3\vartheta}=$

$$= \int_{-\pi/4}^{0} \frac{\sin\vartheta}{\cos^4\vartheta}\,d\vartheta - 2\int_{-\pi/4}^{0} \frac{d\vartheta}{\cos^3\vartheta} = I_1 - 2I_2$$

To evaluate I_1, let $y = \cos\vartheta$. Then

$dy = (\cos\vartheta)'\,d\vartheta = -\sin\vartheta\,d\vartheta \Rightarrow \sin\vartheta\,d\vartheta = -dy.$

and

$\vartheta = -\pi/4 \Rightarrow y = \cos(-\pi/4) = \cos(\pi/4) = \sqrt{2}/2$

$\vartheta = 0 \Rightarrow y = \cos(0) = 1$

therefore

$$I_1 = \int_{-\pi/4}^{0} \frac{\sin\vartheta}{\cos^4\vartheta}\,d\vartheta = \int_{\sqrt{2}/2}^{1} \frac{-dy}{y^4} = \int_{\sqrt{2}/2}^{1} (-y^{-4})\,dy =$$

$$= \left[-\frac{y^{-3}}{-3}\right]_{\sqrt{2}/2}^{1} = \left[\frac{1}{3y^3}\right]_{\sqrt{2}/2}^{1} =$$

$$= \frac{1}{3} - \frac{1}{3(\sqrt{2}/2)^3} = \frac{1}{3} - \frac{1}{3(2\sqrt{2}/8)} =$$

$$= \frac{1}{3}\left[1 - \frac{8}{2\sqrt{2}}\right] = \frac{1}{3}\left[1 - \frac{4}{\sqrt{2}}\right] =$$

$$= \frac{1}{3}\left[1 - \frac{4\sqrt{2}}{2}\right] = \frac{1-2\sqrt{2}}{3}$$

To evaluate I_2, we write

$$I_2 = \int_{-\pi/4}^{0} \frac{d\vartheta}{\cos^3\vartheta} = \int_{-\pi/4}^{0} \frac{\cos\vartheta\,d\vartheta}{\cos^4\vartheta} = \int_{-\pi/4}^{0} \frac{\cos\vartheta\,d\vartheta}{(1-\sin^2\vartheta)^2}$$

Let $y = \sin\vartheta$. Then $dy = (\sin\vartheta)'\,d\vartheta = \cos\vartheta\,d\vartheta$

For $\vartheta = -\pi/4 \Rightarrow y = \sin(-\pi/4) = -\sin(\pi/4) = -\sqrt{2}/2$

For $\vartheta = 0 \Rightarrow y = \sin 0 = 0$

It follows that

$$I_2 = \int_{-\sqrt{2}/2}^{0} \frac{dy}{(1-y^2)^2} = \int_{-\sqrt{2}/2}^{0} \frac{dy}{(1-y)^2(1+y)^2} =$$

$$= \int_{-\sqrt{2}/2}^{0} \frac{dy}{(y-1)^2(y+1)^2} =$$

$$= \int_{-\sqrt{2}/2}^{0} \left[\frac{A}{(y-1)^2} + \frac{B}{y-1} + \frac{C}{(y+1)^2} + \frac{D}{y+1} \right] dy$$

with

$$A = \left. \frac{1}{(y+1)^2} \right|_{y=1} = \frac{1}{(1+1)^2} = \frac{1}{4}$$

$$B = \left. \frac{d}{dy} \left[\frac{1}{(y+1)^2} \right] \right|_{y=1} = \left. \frac{d}{dy} (y+1)^{-2} \right|_{y=1} =$$

$$= \left. -2(y+1)^{-3} \right|_{y=1} = \left. \frac{-2}{(y+1)^3} \right|_{y=1} = \frac{-2}{(1+1)^3}$$

$$= \frac{-2}{8} = \frac{-1}{4}$$

$$C = \left. \frac{1}{(y-1)^2} \right|_{y=-1} = \frac{1}{(-1-1)^2} = \frac{1}{(-2)^2} = \frac{1}{4}$$

$$D = \left. \frac{d}{dy} \frac{1}{(y-1)^2} \right|_{y=-1} = \left. \frac{d}{dy} (y-1)^{-2} \right|_{y=-1} =$$

$$= \left. -2(y-1)^{-3} \right|_{y=-1} = \left. \frac{-2}{(y-1)^3} \right|_{y=-1} = \frac{-2}{(-1-1)^3}$$

$$= \frac{-2}{-8} = \frac{1}{4}$$

and therefore,

$$I_2 = \int_{-\sqrt{2}/2}^{0} \left[\frac{1/4}{(y-1)^2} + \frac{-1/4}{y-1} + \frac{1/4}{(y+1)^2} + \frac{1/4}{y+1} \right] dy =$$

$$= \frac{1}{4} \left[\frac{-1}{y-1} - \ln|y-1| + \frac{-1}{y+1} + \ln|y+1| \right]_{-\sqrt{2}/2}^{0} =$$

$$= \frac{1}{4} \left[\frac{-(y+1)-(y-1)}{(y-1)(y+1)} + \ln\left| \frac{y+1}{y-1} \right| \right]_{-\sqrt{2}/2}^{0}$$

$$= \frac{1}{4} \left[\frac{-2y}{y^2-1} + \ln\left| \frac{y+1}{y-1} \right| \right]_{-\sqrt{2}/2}^{0} =$$

$$= \frac{1}{4} \left[0 + \ln\left| \frac{0+1}{0-1} \right| \right] -$$

$$- \frac{1}{4} \left[\frac{-2(-\sqrt{2}/2)}{(-\sqrt{2}/2)^2 - 1} + \ln\left| \frac{-\sqrt{2}/2+1}{-\sqrt{2}/2-1} \right| \right]$$

$$= (1/4)(0+\ln 1) - \frac{1}{4} \left[\frac{\sqrt{2}}{1/2-1} + \ln\left| \frac{\sqrt{2}-2}{\sqrt{2}+2} \right| \right]$$

$$= \frac{-1}{4} \left[\frac{\sqrt{2}}{-1/2} + \ln\left| \frac{(\sqrt{2}-2)^2}{(\sqrt{2}+2)(\sqrt{2}-2)} \right| \right]$$

$$= \frac{-1}{4} \left[-2\sqrt{2} + \ln\left| \frac{2-4\sqrt{2}+4}{2-4} \right| \right]$$

$$= \frac{-1}{4} \left[-2\sqrt{2} + \ln\left| \frac{6-4\sqrt{2}}{-2} \right| \right] =$$

$$= \frac{+\sqrt{2}}{2} - \frac{\ln(3-2\sqrt{2})}{4}$$

We conclude that

$$I = I_1 - 2I_2 = \frac{1-2\sqrt{2}}{3} - 2\left[\frac{\sqrt{2}}{2} - \frac{\ln(3-2\sqrt{2})}{4} \right]$$

$$= \frac{1-2\sqrt{2}}{3} - \sqrt{2} + \frac{\ln(3-2\sqrt{2})}{2} = \frac{1-5\sqrt{2}}{3} + \frac{\ln(3-2\sqrt{2})}{2}$$

EXERCISE

(41) Evaluate the following integrals

a) $I = \int_1^2 \frac{\sqrt{x^2+9}}{x}\,dx$

b) $I = \int_0^2 \sqrt{x^2+4}\,dx$

c) $I = \int_0^1 x^3\sqrt{x^2+1}\,dx$

d) $I = \int \frac{x^2\,dx}{\sqrt{x^2+7}}$

e) $I = \int \frac{dx}{x^2\sqrt{x^2+1}}$

f) $I = \int_0^1 x^3\sqrt{x^2+x+2}\,dx$

CAL 2.3: Improper Integrals

IMPROPER INTEGRALS

Improper integrals of the first kind

Def : Let $f: A \to \mathbb{R}$ be a function such that
$\forall b \in (a, +\infty)$: f integrable on $[a, b]$
Then, we define:

$$\boxed{\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx}$$

Def : Let $f: A \to \mathbb{R}$ be a function such that
$\forall a \in (-\infty, b)$: f integrable on $[a, b]$
Then, we define

$$\boxed{\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx}$$

Def : Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that
$\forall a, b \in \mathbb{R}$: $(a < b \Rightarrow f$ integrable on $[a, b])$
Then, we define:

$$\boxed{\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{+\infty} f(x)\,dx}$$

↳ In all of the above cases, we say that the corresponding improper integral converges if and only if the corresponding limit is equal to a real number. Otherwise, we say that the improper integral diverges.

↳ Notation

a) For $S = [a, +\infty)$: $\int_S f(x)\,dx = \int_a^{+\infty} f(x)\,dx$

b) For $S = (-\infty, b]$: $\int_S f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx$

c) For $S = [a, b]$: $\int_S f(x)\,dx = \int_a^b f(x)\,dx$

d) Likewise, $\int_{\mathbb{R}} f(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{+\infty} f(x)\,dx$

- Simple improper integrals can be evaluated by calculating the corresponding finite integral and then taking the limit.
- The integral over $\mathbb{R}$ should be distinguished from the principle value integral over $\mathbb{R}$ which is defined as

$$\text{PV}\int_{\mathbb{R}} f(x)\,dx = \lim_{a\to+\infty} \int_{-a}^{a} f(x)\,dx$$

Note that the integral over $\mathbb{R}$ can diverge but the principle value integral over $\mathbb{R}$ can converge.

EXAMPLE

a) Evaluate the integral $I=\int_0^{+\infty} xe^{-x}dx$

Solution

We define:

$$f(x)=\int_0^x te^{-t}dt=\int_0^x(-t)(-e^{-t})dt=\int_0^x(-t)(e^{-t})'dt=$$

$$=\left[-te^{-t}\right]_0^x-\int_0^x(-t)'e^{-t}dt=$$

$$=-\left[xe^{-x}-0e^{-0}\right]+\int_0^x e^{-t}dt=-xe^{-x}+\left[-e^{-t}\right]_0^x=$$

$$=-xe^{-x}-\left[e^{-x}-e^{0}\right]=-xe^{-x}-\left[e^{-x}-1\right]=1-e^{-x}-xe^{-x}$$

$$=1-(x+1)e^{-x},\ \forall x\in(0,+\infty)\Rightarrow$$

$$\Rightarrow I=\int_0^{+\infty} xe^{-x}dx=\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\left[1-(x+1)e^{-x}\right]=$$

$$=1-\lim_{x\to+\infty}\frac{x+1}{e^x}=1-\lim_{x\to+\infty}\frac{(x+1)'}{(e^x)'}=$$

$$=1-\lim_{x\to+\infty}\frac{1}{e^x}=1-0=1.$$

b) Evaluate the integral $I=\int_2^{+\infty}\frac{dx}{x\ln x}$

Solution

Define $f(t)=\int_2^t\frac{dx}{x\ln x}$, $\forall t\in(2,+\infty)$ and let

$y = \ln x \Rightarrow dy = dx/x$

It follows that

$$f(t) = \int_{\ln 2}^{\ln t} \frac{dy}{y} = \Big[\ln|y|\Big]_{\ln 2}^{\ln t} = \ln|\ln t| - \ln|\ln 2|$$

$$= \ln(\ln t) - \ln(\ln 2), \quad \forall t \in (2, +\infty)$$

Since

$$\lim_{t\to+\infty} \ln t = +\infty \Rightarrow \lim_{t\to+\infty} \ln(\ln t) = +\infty \Rightarrow$$

$$\Rightarrow I = \int_0^{+\infty} \frac{dx}{x\ln x} = \lim_{t\to+\infty} \int_0^t \frac{dx}{x\ln x} = \lim_{t\to+\infty} f(t) =$$

$$= \lim_{t\to+\infty} [\ln(\ln t) - \ln(\ln 2)] = +\infty \Rightarrow$$

$$\Rightarrow \int_0^{+\infty} \frac{dx}{x\ln x} \text{ diverges.}$$

c) Show that $\int_{\mathbb{R}} x^3 dx$ diverges, but $\oint_{\mathbb{R}} x^3 dx = 0$

Solution

Since,

$$\int_0^{+\infty} x^3 dx = \lim_{a \to +\infty} \int_0^a x^3 dx = \lim_{a \to +\infty} \left[\frac{x^4}{4} \right]_0^a = \lim_{a \to +\infty} \frac{a^4}{4}$$

$$= +\infty \Rightarrow \int_0^{+\infty} x^3 dx \text{ diverges} \Rightarrow$$

$$\Rightarrow \int_{\mathbb{R}} x^3 dx \text{ diverges.}$$

but

$$\oint_{\mathbb{R}} x^3 dx = \lim_{a \to +\infty} \int_{-a}^a x^3 dx = \lim_{a \to +\infty} \left[\frac{x^4}{4} \right]_{-a}^a =$$

$$= \lim_{a \to +\infty} \frac{a^4 - (-a)^4}{4} = \lim_{a \to +\infty} 0 = 0$$

EXERCISES

(1) Evaluate the following integrals

a) $I = \int_0^{+\infty} x \exp(-x^2)\,dx$

b) $I = \int_2^{+\infty} \frac{dx}{x^2+4}$

c) $I = \int_1^{+\infty} \frac{x\,dx}{(1+x^2)^2}$

d) $I = \int_0^{+\infty} x^2 e^{-x}\,dx$

e) $I = \int_0^{+\infty} e^{-x}\cos(2x)\,dx$

f) $I = \int_{-\infty}^{+\infty} \frac{dx}{x^2+2x+10}$

g) $I = \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2}$

h) $I = \int_1^{+\infty} \frac{dx}{x\sqrt{x^2+1}}$

i) $I = \int_2^{+\infty} \frac{dx}{x\sqrt{x^2-1}}$

j) $I = \int_2^{+\infty} \frac{dx}{x^2\sqrt{x^2-1}}$

k) $I = \int_1^{+\infty} \frac{\ln x}{x^2}\,dx$

l) $I = \int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+9}}\,dx$

Improper integrals of the second kind

Def: Let $f: A \to \mathbb{R}$ be a function such that
$\forall t \in (a,b)$: f integrable on $[t,b]$
Then we define

$$\int_{a^+}^{b} f(x)dx = \lim_{t \to a^+} \int_{t}^{b} f(x)dx$$

Def: Let $f: A \to \mathbb{R}$ be a function such that
$\forall t \in (a,b)$: f integrable on $[a,t]$
Then, we define

$$\int_{a}^{b^-} f(x)dx = \lim_{t \to b^-} \int_{a}^{t} f(x)dx$$

Def: Let $f: A \to \mathbb{R}$ be a function such that
$\forall s,t \in (a,b)$: $(s<t \Rightarrow f$ integrable on $[s,t])$
and let $c \in (a,b)$ be given. Then, we define:

$$\int_{a^+}^{b^-} f(x)dx = \int_{a^+}^{c} f(x)dx + \int_{c}^{b^-} f(x)dx$$

↳ It is easy to see that the last definition is independent of the choice of $c \in (a,b)$.

↳ We extend the definition of the notation

$$I = \int_{\mathcal{S}} f(x)\,dx$$

where $\mathcal{S} = (a,b]$ or $\mathcal{S} = [a,b)$ or $\mathcal{S} = (a,b)$ as follows:

$$\int_{(a,b]} f(x)\,dx = \int_{a^+}^{b} f(x)\,dx$$

$$\int_{[a,b)} f(x)\,dx = \int_{a}^{b^-} f(x)\,dx$$

$$\int_{(a,b)} f(x)\,dx = \int_{a^+}^{b^-} f(x)\,dx$$

EXAMPLES

a) $I = \int_0^{1^-} \frac{dx}{\sqrt{1-x^2}}$

Solution

Since

$$f(t) = \int_0^t \frac{dx}{\sqrt{1-x^2}} = [\text{Arcsin}\, x]_0^t = \text{Arcsin}(t) - \text{Arcsin}(0) =$$

$$= \text{Arcsin}(t) - 0 = \text{Arcsin}(t) \Rightarrow$$

$$\Rightarrow I = \int_0^{1^-} \frac{dx}{\sqrt{1-x^2}} = \lim_{t \to 1^-} f(t) = \lim_{t \to 1^-} \text{Arcsin}(t) =$$

$$= \text{Arcsin}(1) = \pi/2$$

b) $I = \int_{-(\pi/2)^+}^{(\pi/2)^-} \tan x \, dx$

Solution

Define $f(\$, t) = \int_{\$}^{t} \tan x \, dx$, $\forall \$, t \in (-\pi/2, \pi/2)$

and let $y = \tan x$. It follows that

$$dy = (1 + \tan^2 x)\, dx = (1 + y^2)\, dx \Rightarrow dx = \frac{dy}{1+y^2}$$

and therefore:

$$f(s,t) = \int_{\tan(s)}^{\tan(t)} y \frac{dy}{1+y^2} = \frac{1}{2}\int_{\tan(s)}^{\tan(t)} \frac{2y}{1+y^2}\,dy =$$

$$= \frac{1}{2}\int_{\tan(s)}^{\tan(t)} \frac{(1+y^2)'}{1+y^2}\,dy = \frac{1}{2}\int_{\tan(s)}^{\tan(t)} [\ln(1+y^2)]'\,dy$$

$$= \frac{1}{2}\left[\ln(1+\tan^2 t) - \ln(1+\tan^2 s)\right] =$$

$$= \frac{1}{2}\ln\left(\frac{1+\tan^2 t}{1+\tan^2 s}\right)$$

Let us consider

$$\int_0^{(\pi/2)^-} \tan x\,dx = \lim_{t\to(\pi/2)^-} f(0,t)$$

Since $f(0,t) = \frac{1}{2}\ln\left(\frac{1+\tan^2 t}{1+\tan^2 0}\right) = \frac{1}{2}\ln\left(\frac{1+\tan^2 t}{1+0}\right) =$

$$= \frac{1}{2}\ln(1+\tan^2 t)$$

and

$$\lim_{t\to(\pi/2)^-} \tan t = +\infty \Rightarrow \lim_{t\to(\pi/2)^-} \tan^2 t = +\infty \Rightarrow$$

$$\Rightarrow \lim_{t\to(\pi/2)^-} (1+\tan^2 t) = +\infty \Rightarrow$$

$$\Rightarrow \lim_{t\to(\pi/2)^-} \ln(1+\tan^2 t) = +\infty \Rightarrow$$

$$\Rightarrow \int_0^{(\pi/2)^-} \tan x\,dx = \lim_{t\to(\pi/2)^-} f(0,t) = +\infty \Rightarrow \int_0^{(\pi/2)^-} \tan x\,dx \text{ diverges}$$

$$\Rightarrow \int_{-(\pi/2)^+}^{(\pi/2)^-} \tan x\,dx \text{ diverges.}$$

EXERCISES

(2) Evaluate the following integrals or show they diverge

a) $I = \int_0^{3^-} \frac{dx}{\sqrt{3-x}}$

b) $I = \int_{1^+}^{2} \frac{dx}{(x-1)^{4/3}}$

c) $I = \int_0^{2^-} \frac{x\,dx}{4-x^2}$

d) $I = \int_0^{1^-} \frac{3x\,dx}{\sqrt{1-x^2}}$

e) $I = \int_{-1^+}^{0} \frac{dx}{x^2+3x+2}$

f) $I = \int_0^{1^-} \frac{dx}{x^3-x^2-x+1}$

g) $I = \int_0^{(\pi/6)^-} \tan(3x)\,dx$

h) $I = \int_{0^+}^{\pi/4} \frac{dx}{\sin x}$

i) $I = \int_0^{(\pi/2)^-} \frac{dx}{\cos^2 x}$

j) $I = \int_{\pi/4}^{(\pi/2)^-} \frac{\sin x}{1-\cos x}\,dx$

k) $I = \int_1^{e^-} \frac{dx}{x \ln x}$

l) $I = \int_1^{e^-} \frac{dx}{x(\ln x)^3}$

m) $I = \int_0^{(\pi/2)^-} \frac{dx}{1-\sin x}$

n) $I = \int_{0^+}^{1} \frac{e^x}{\sqrt{e^x-1}}\,dx$

Convergence criteria for improper integrals

- It is often possible to determine if an improper integral converges or diverges without evaluating it, by comparing it with other known integrals.
- To state the convergence criteria concisely, let $S \subseteq \mathbb{R}$ be the range of integration of an improper integral defined as a limit of the form

$$\int_S f(x)dx = \lim_{t \to \sigma} \int_{S(t)} f(x)dx$$

with $\sigma = a^+$ or $\sigma = a^-$ (with $a \in \mathbb{R}$) or $\sigma = +\infty$ or $\sigma = -\infty$ and with $S(t) \subseteq S$ where $S(t)$ is a closed interval. There are 4 possibilities:

S	$[a, +\infty)$	$(-\infty, b]$	$(a, b]$	$[a, b)$
$S(t)$	$[a, t]$	$[t, b]$	$[t, b]$	$[a, t]$
$t \to \sigma$	$+\infty$	$-\infty$	a^+	b^-

- Also recall that given any statement of the form $p \Rightarrow q$, proving this statement also proves the contrapositive statement $\bar{q} \Rightarrow \bar{p}$, where $\bar{p}, \bar{q}$ are the negations of p, q.

Convergence criteria

(1) Comparison test

$$\left.\begin{array}{l} \forall x \in \$: 0 \leq f(x) \leq g(x) \\ \int_{\$} g(x)\,dx \text{ converges} \end{array}\right\} \Rightarrow \int_{\$} f(x)\,dx \text{ converges}$$

$$\left.\begin{array}{l} \forall x \in \$: 0 \leq f(x) \leq g(x) \\ \int_{\$} f(x)\,dx \text{ diverges} \end{array}\right\} \Rightarrow \int_{\$} g(x)\,dx \text{ diverges}$$

↳ Note that the second statement can be obtained as a contrapositive of the first statement.

(2) Ratio test

$$\left.\begin{array}{l} \forall x \in \$: (f(x) \geq 0 \wedge g(x) \geq 0) \\ \lim\limits_{x \to \sigma} \dfrac{f(x)}{g(x)} \in (0, +\infty) \end{array}\right\} \Rightarrow \left(\int_S f(x)\,dx \text{ converges} \Leftrightarrow \int_S g(x)\,dx \text{ converges} \right)$$

$$\left.\begin{array}{l} \forall x \in \$: (f(x) \geq 0 \wedge g(x) \geq 0) \\ \lim\limits_{x \to \sigma} \dfrac{f(x)}{g(x)} = 0 \end{array}\right\} \Rightarrow \left(\int_S g(x)\,dx \text{ converges} \Rightarrow \int_S f(x)\,dx \text{ converges} \right)$$

$$\left.\begin{array}{l} \forall x \in \$: (f(x) \geq 0 \wedge g(x) \geq 0) \\ \lim\limits_{x \to \sigma} \dfrac{f(x)}{g(x)} = +\infty \end{array}\right\} \Rightarrow \left(\int_{\$} f(x)\,dx \text{ converges} \Rightarrow \int_S g(x)\,dx \text{ converges} \right)$$

↳ The corresponding contrapositives statements are:

$$\left.\begin{array}{l}\forall x\in\$:(f(x)\geq 0 \wedge g(x)>0)\\ \lim\limits_{x\to\sigma}\dfrac{f(x)}{g(x)}\in(0,+\infty)\end{array}\right\}\Rightarrow\left(\int_{\$}f(x)dx \text{ diverges}\Leftrightarrow\int_{\$}g(x)dx \text{ diverges}\right)$$

$$\left.\begin{array}{l}\forall x\in\$:(f(x)\geq 0 \wedge g(x)\geq 0)\\ \lim\limits_{x\to\sigma}\dfrac{f(x)}{g(x)}=0\end{array}\right\}\Rightarrow\left(\int_{\$}f(x)dx \text{ diverges}\Rightarrow\int_{\$}g(x)dx \text{ diverges}\right)$$

$$\left.\begin{array}{l}\forall x\in\$:(f(x)\geq 0 \wedge g(x)\geq 0)\\ \lim\limits_{x\to\sigma}\dfrac{f(x)}{g(x)}=+\infty\end{array}\right\}\Rightarrow\left(\int_{\$}g(x)dx \text{ diverges}\Rightarrow\int_{\$}f(x)dx \text{ diverges}\right)$$

↳ Note that the limit calculation can be interpreted as a comparison of <u>growth rates</u> of $f(x)$ and $g(x)$ as $x\to\sigma$. Specifically we can give the following definitions:

a) $f(x)$ and $g(x)$ have equivalent growth rates as $x\to\sigma$:

$$f(x)\sim g(x) \text{ as } x\to\sigma \Leftrightarrow \lim_{x\to\sigma}\frac{f(x)}{g(x)}=1$$

b) $f(x)$ has a slower rate of growth than $g(x)$ as $x\to\sigma$:

$$f(x)\ll g(x) \text{ as } x\to\sigma \Leftrightarrow \lim_{x\to\sigma}\frac{f(x)}{g(x)}=0$$

The interpretation of the ratio test becomes obvious when you realize that convergence requires a sufficiently small growth rate.

③ Absolute convergence

$$\int_S |f(x)|\,dx \text{ converges} \Rightarrow \int_S f(x)\,dx \text{ converges}$$

↳ The corresponding contrapositive statement is:

$$\int_S f(x)\,dx \text{ diverges} \Rightarrow \int_S |f(x)|\,dx \text{ diverges.}$$

↳ Note that the absolute convergence criterio is often used in conjunction with the comparison test or with the ratio test.

↳ Special Improper integrals

1) $I = \int_a^{+\infty} \frac{dx}{x^p}$ converges $\Leftrightarrow p > 1$ (with $a \in (0, +\infty)$)

2) $I = \int_a^{+\infty} e^{-px}\,dx$ converges $\Leftrightarrow p > 0$ (with $a \in \mathbb{R}$)

3) $I = \int_{a^+}^{b} \frac{dx}{(x-a)^p}$ converges $\Leftrightarrow p < 1$ (with $b \in (a, +\infty)$)

4) $I = \int_a^{b^-} \frac{dx}{(b-x)^p}$ converges $\Leftrightarrow p < 1$ (with $a \in (-\infty, b)$)

These results can be established by direct integral evaluation.

EXAMPLES

a) $I = \int_1^{+\infty} \frac{x\,dx}{3x^4+2x^2+3}$ ← determine if I is convergent

Solution

1st method: Via Comparison test.

We claim that $\forall x \in (1,+\infty)$: $\frac{x}{3x^4+2x^2+3} \leq \frac{1}{3x^3}$ (1)

Let $x \in (1,+\infty)$ be given. Then:

$x>1 \Rightarrow x^2>1^2 \Rightarrow 2x^2>2 \Rightarrow 2x^2+3>2+3=5 \Rightarrow$

$\Rightarrow 3x^4+2x^2+3 > 5+3x^4 > 3x^4 > 0 \Rightarrow$

$\Rightarrow 3x^4+2x^2+3 > 3x^4 > 0 \Rightarrow$

$\Rightarrow 0 < \frac{1}{3x^4+2x^2+3} < \frac{1}{3x^4} \Rightarrow$

$\Rightarrow \frac{x}{3x^4+2x^2+3} < \frac{x}{3x^4}$ [because $x>0$]

$\Rightarrow \frac{x}{3x^4+2x^2+3} < \frac{1}{3x^3}$

and this proves the claim.

We also note that

$\int_1^{+\infty} \frac{dx}{x^3}$ converges $\Rightarrow \int_1^{+\infty} \frac{dx}{3x^3}$ converges. (2)

From Eq.(1) and Eq.(2) via the comparison test, it follows that I converges.

2nd method : Via ratio test.

We define :

$$f(x) = \frac{x}{3x^4+2x^2+3}, \quad \forall x \in (1,+\infty)$$

$$g(x) = \frac{1}{x^3}, \quad \forall x \in (1,+\infty)$$

It follows that

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{\left(\frac{x}{3x^4+2x^2+3}\right)}{\left(\frac{1}{x^3}\right)} = \lim_{x\to+\infty} \frac{x^4}{3x^4+2x^2+3}$$

$$= \lim_{x\to+\infty} \frac{x^4}{3x^4} = \frac{1}{3} \qquad (1)$$

We also note that $\int_1^{+\infty} g(x)\,dx = \int_1^{+\infty} \frac{dx}{x^3}$ converges (2)

From Eq.(1) and Eq.(2), via the ratio test, it follows that $I = \int_1^{+\infty} f(x)\,dx$ converges.

↳ Note that the key insight in both methods is to recognize that the integrand has the same growth rate as $1/x^3$ in the limit $x \to +\infty$. Also note that establishing the inequality, when using the comparison test, can be challenging, and requires a careful proof.

B) Show that $I = \int_0^{+\infty} e^{-x^2}\,dx$ converges.

Solution

1st method: Ratio test

We define:

$f(x) = \exp(-x^2), \ \forall x \in (1, +\infty)$

$g(x) = 1/x^2, \ \forall x \in (1, +\infty)$

and note that

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{\exp(-x^2)}{1/x^2} = \lim_{x\to+\infty} x^2 \exp(-x^2) =$$

$$= \lim_{x\to+\infty} \frac{x^2}{\exp(x^2)} = \lim_{x\to+\infty} \frac{(x^2)'}{[\exp(x^2)]'} = \lim_{x\to+\infty} \frac{2x}{(x^2)'\exp(x^2)}$$

$$= \lim_{x\to+\infty} \frac{2x}{2x\cdot\exp(x^2)} = \lim_{x\to+\infty} \frac{1}{\exp(x^2)} = 0 \qquad (1)$$

because $\lim_{x\to+\infty} x^2 = +\infty \Rightarrow \lim_{x\to+\infty} \exp(x^2) = +\infty \Rightarrow \lim_{x\to+\infty} \frac{1}{\exp(x^2)} = 0$

We also note that $\int_1^{+\infty} g(x)dx = \int_1^{+\infty} \frac{dx}{x^2}$ converges (2)

From Eq. (1) and Eq. (2), via the ratio test, it follows that

$$\int_1^{+\infty} e^{-x^2}dx \text{ converges} \Rightarrow \int_0^{+\infty} e^{-x^2}dx \text{ converges.}$$

↳ Note that for our convenience, we may restrict the range of integration in order to establish convergence.

↳ This problem can be also done via the comparison test, using the following fundamental inequality from my Calculus 1 Lecture Notes:

$\forall x \in \mathbb{R}: \; e^x \geq x+1$

2nd method : Comparison test.

Let $x \in [1, +\infty)$ be given. Then:

$$e^{x^2} \geq x^2+1 > x^2 > 0 \Rightarrow \frac{1}{e^{x^2}} < \frac{1}{x^2} \Rightarrow e^{-x^2} < \frac{1}{x^2}$$

It follows that $e^{-x^2} < \frac{1}{x^2}, \; \forall x \in [1, +\infty)$ (1)

We also note that

$$\int_1^{+\infty} \frac{dx}{x^2} \quad \text{converges} \qquad (2)$$

From Eq.(1) and Eq.(2), via the comparison test, we have:

$$\int_1^{+\infty} \exp(-x^2)\,dx \quad \text{converges} \Rightarrow \int_0^{+\infty} \exp(-x^2)\,dx \quad \text{converges.}$$

↳ In general, the integral of $\exp(-x^2)$ cannot be evaluated in terms of elementary functions. It is in fact used to define a completely new function, the Gauss error function $\operatorname{erf}(x)$ given by:

$$\boxed{\forall x \in \mathbb{R}: \; \operatorname{erf}(x) = \int_0^x \frac{2\exp(-t^2)\,dt}{\sqrt{\pi}}}$$

It can be shown that:

$$\lim_{x \to -\infty} \operatorname{erf}(x) = -1 \quad \text{and} \quad \lim_{x \to +\infty} \operatorname{erf}(x) = 1.$$

c) Show that $I = \int_1^{+\infty} \frac{\ln x}{2x+3}\, dx$ diverges.

Solution

We define : $f(x) = \frac{\ln x}{2x+3}$, $\forall x \in (1, +\infty)$

$$g(x) = \frac{1}{x} \quad , \forall x \in (1, +\infty)$$

and note that

$$\forall x \in (1,+\infty): \frac{f(x)}{g(x)} = \frac{\left(\frac{\ln x}{2x+3}\right)}{\left(\frac{1}{x}\right)} = \frac{x \ln x}{2x+3} \qquad (1)$$

and

$$\lim_{x \to +\infty} \frac{x}{2x+3} = \lim_{x \to +\infty} \frac{x}{2x} = \frac{1}{2} \qquad (2)$$

and $$\lim_{x \to +\infty} \ln x = +\infty \qquad (3)$$

From Eq.(1) and Eq.(2) and Eq.(3) : $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = +\infty$ (4)

We also note that $\int_1^{+\infty} \frac{dx}{x}$ diverges. (5)

From Eq.(4) and Eq.(5), via the ratio test, it follows that

$$I = \int_1^{+\infty} \frac{\ln x}{2x+3}\, dx \quad \text{diverges.}$$

d) Show that $I = \int_{0^+}^{1} \frac{dx}{x^2(x^2+3)}$ diverges.

Solution

↳ Note that as $x \to 0^+$, in the denominator, the dominant term is the lowest-order term $3x^2$. Consequently, we must compare the integrand with $1/x^2$.

1st method : Ratio test

We define:

$$f(x) = \frac{1}{x^2(x^2+3)} = \frac{1}{x^4+3x^2}, \quad \forall x \in (0,1]$$

$$g(x) = \frac{1}{x^2}, \quad \forall x \in (0,1]$$

and therefore,

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\frac{1}{x^2(x^2+3)}}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{x^2}{x^2(x^2+3)} =$$

$$= \lim_{x\to 0^+} \frac{1}{x^2+3} = \frac{1}{0^2+3} = \frac{1}{3} \qquad (1)$$

From Eq.(1), via the ratio test, it follows that

$$\int_{0^+}^{1} \frac{dx}{x^2} \text{ diverges} \Rightarrow I = \int_{0^+}^{1} \frac{dx}{x^2(x^2+3)} \text{ diverges}$$

2nd method : Comparison test

Let $x \in (0,1]$ be given. Since,

$\begin{cases} 0<x<1 \\ 2<4 \end{cases} \Rightarrow x^2 > x^4$, and therefore,

$x^2(x^2+3) = x^4+3x^2 < x^2+3x^2 = 4x^2 \qquad (1)$

We also have:

$x>0 \Rightarrow \begin{cases} x^2>0 \\ x^2+3>0 \end{cases} \Rightarrow x^2(x^2+3)>0 \qquad (2)$

From Eq.(1) and Eq.(2):

$0 < x^2(x^2+3) < 4x^2 \Rightarrow \dfrac{1}{x^2(x^2+3)} > \dfrac{1}{4x^2}$

It follows that $\forall x \in (0,1]: \dfrac{1}{x^2(x^2+3)} > \dfrac{1}{4x^2} \qquad (3)$

From Eq.(3), via the comparison test, it follows that

$\int_{0+}^{1} \dfrac{dx}{x^2}$ diverges $\Rightarrow \int_{0+}^{1} \dfrac{dx}{4x^2}$ diverges

$\Rightarrow \int_{0+}^{1} \dfrac{dx}{x^2(x^2+3)}$ diverges

e) Show that $I = \int_1^{+\infty} \frac{\sin(\pi x)}{\pi x}\, dx$ converges.

↳ It may seem that the integrand scales as $1/x$ and may therefore diverge. However, using integration by parts we can get growth rate $1/x^2$. Note that this problem also uses the absolute convergence theorem and the zero-bounded theorem.

Solution

We define

$$f(t) = \int_1^t \frac{\sin(\pi x)}{\pi x}\, dx = \int_1^t \left(\frac{-\cos(\pi x)}{\pi}\right)' \frac{1}{\pi x}\, dx =$$

$$= \left[\frac{-\cos(\pi x)}{\pi}\frac{1}{\pi x}\right]_1^t - \int_1^t \left(\frac{-\cos(\pi x)}{\pi}\right)\left(\frac{1}{\pi x}\right)' dx$$

$$= -\left[\frac{\cos(\pi t)}{\pi^2 t} - \frac{\cos(\pi)}{\pi^2}\right] - \int_1^t \left(\frac{-\cos(\pi x)}{\pi}\right)\left(\frac{-1}{\pi x^2}\right) dx$$

$$= \frac{-1}{\pi^2} - \frac{\cos(\pi t)}{\pi^2 t} - \int_1^t \frac{\cos(\pi x)}{(\pi x)^2}\, dx \qquad (1)$$

We need to show that all terms converge as $t \to +\infty$.

From the zero-bounded theorem (see my Calculus 1 online lecture notes)

$$\begin{cases} \forall t \in (1, +\infty): |\cos(\pi t)| \leq 1 \\ \lim_{t\to+\infty} \frac{1}{\pi^2 t} = 0 \end{cases} \Rightarrow \lim_{t\to+\infty} \frac{\cos(\pi t)}{\pi^2 t} = 0 \qquad (2)$$

We also note that:

$$\forall x \in (1, +\infty): \left|\frac{\cos(\pi x)}{(\pi x)^2}\right| = \frac{|\cos(\pi x)|}{|(\pi x)^2|} = \frac{|\cos(\pi x)|}{(\pi x)^2} \leq \frac{1}{(\pi x)^2}$$

and it follows that

$$\int_1^{+\infty} \frac{dx}{x^2} \text{ converges} \Rightarrow \int_1^{+\infty} \frac{dx}{(\pi x)^2} \text{ converges} \Rightarrow$$

$$\Rightarrow \int_1^{+\infty} \left| \frac{\cos(\pi x)}{(\pi x)^2} \right| dx \text{ converges} \Rightarrow$$

$$\Rightarrow \int_1^{+\infty} \frac{\cos(\pi x)}{(\pi x)^2} dx \text{ converges.}$$

From Eq. (1) and Eq. (2):

$$I = \int_1^{+\infty} \frac{\sin(\pi x)}{\pi x} dx =$$

$$= \frac{-1}{\pi^2} - \lim_{t \to +\infty} \frac{\cos(\pi t)}{\pi^2 t} - \int_1^{+\infty} \frac{\cos(\pi x)}{(\pi x)^2} dx$$

$$= \frac{-1}{\pi^2} - \int_1^{+\infty} \frac{\cos(\pi x)}{(\pi x)^2} dx \Rightarrow \int_1^{+\infty} \frac{\sin(\pi x)}{\pi x} dx \text{ converges.}$$

EXERCISES

(3) Without evaluating the following integrals, show whether they converge or diverge

a) $I = \int_0^{+\infty} \frac{3x\,dx}{x^5+x+1}$

b) $I = \int_0^{+\infty} x^p e^{-x^2}\,dx$, with $p \geq 0$.

c) $I = \int_0^{+\infty} \frac{x\,dx}{\sqrt{x^4+3}}$

d) $I = \int_2^{+\infty} \frac{dx}{x^2 \ln x}$

e) $I = \int_1^{+\infty} \frac{dx}{\sqrt{x}+\sqrt{x+1}}$

f) $I = \int_0^{+\infty} \frac{dx}{\sqrt{x^3+2}}$

g) $I = \int_e^{+\infty} \frac{dx}{\sqrt{x}\ln x}$

h) $I = \int_3^{+\infty} \frac{dx}{(x \ln x)^2}$

i) $I = \int_0^{+\infty} \frac{x\,dx}{e^x}$

(4) Similarly, for the following integrals.

a) $I = \int_0^{1^-} \frac{x\,dx}{\sqrt{1-x^3}}$

b) $I = \int_{-1^+}^{0} \frac{3x}{x^5+1}\,dx$

c) $I = \int_1^{4^-} \frac{x\,dx}{\sqrt{x}-2}$

d) $I = \int_{0^+}^{1} \frac{\ln x}{\sqrt{x}}\,dx$

e) $I = \int_{1/2}^{1^-} \frac{\ln x}{x-1}\,dx$

f) $I = \int_{0^+}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

g) $I = \int_{0^+}^{1/2} \frac{\sqrt{x}}{\arcsin x}\,dx$

h) $I = \int_{0^+}^{1} \frac{dx}{x \arctan(x)}$

CAL 2.4: Convergent sequences

SEQUENCES

Preliminaries

- Recall the following set definitions:
  $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (set of natural numbers)
  $\mathbb{N}^* = \{1, 2, 3, \ldots\} = \mathbb{N} - \{0\}$
  $[n] = \{x \in \mathbb{N}^* \mid x \leq n\} = \{1, 2, 3, \ldots, n\}$
  The set of all natural numbers $n > n_0$ with $n_0 \in \mathbb{N}^*$ can be represented as:
  $\mathbb{N}^* - [n_0] = \{x \in \mathbb{N} \mid x > n_0\}$.
- For the theory of sequences and series we make heavy use of quantifiers $\forall$ and $\exists$. (see my Calculus 1 lecture notes).
  Let $p(x)$ be a predicate (i.e. a statement about x which is either true or false depending on the value of x) and let A be a set. Then:
  1) $\forall x \in A : p(x)$ (universal quantifier)
  means: "For all $x \in A$, $p(x)$ is true"
  2) $\exists x \in A : p(x)$ (existential quantifier)
  means: "There exists (at least one) $x \in A$ such that $p(x)$ is true".
  We can now construct nested statements:
  e.g. $\forall x \in A : \exists y \in B : p(x, y)$
  "For all $x \in A$, there is at least one $y \in B$, such that $p(x, y)$ is true".

Sequences and convergent sequences

- A <u>sequence a_n</u> is a list of numbers
$a_0, a_1, a_2, a_3, \dots, a_n, \dots$
written down in a definite order.
- More rigorously, a sequence a_n is a <u>mapping</u> of the form
$$a: n \in \mathbb{N} \longrightarrow a_n \in \mathbb{R}$$
or $$a: n \in \mathbb{N}^* \longrightarrow a_n \in \mathbb{R}$$
- A sequence can be defined:

a) <u>Explicitly</u>: By providing an algebraic expression that yields a_n in terms of n.
e.g. $\forall n \in \mathbb{N}: a_n = n^2 \cdot 2^n$

b) <u>Recursively</u>: By providing an algorithm that gives the next element of a sequence in terms of previous elements
e.g.
$$\begin{cases} a_1 = 1 \wedge a_2 = 1 \\ \forall n \in \mathbb{N}^*: a_{n+2} = a_n + a_{n+1} \end{cases}$$
↳ This example is the famous <u>Fibonacci sequence</u>.

↳ Convergent sequences

Let a_n be a sequence and let $l \in \mathbb{R}$. Then we define:

$$\lim a_n = l \Leftrightarrow \forall \varepsilon > 0: \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: |a_n - l| < \varepsilon$$

a_n convergent $\Leftrightarrow \exists l \in \mathbb{R}: \lim a_n = l$

- These definitions read:

a) "$\lim a_n = l$ if and only if for all real numbers $\varepsilon > 0$ there is at least one $n_0 \in \mathbb{N}^*$ such that for all $n \in \mathbb{N}^* - [n_0]$ (i.e. natural numbers n with $n > n_0$) we have $|a_n - l| < \varepsilon$"

b) "a_n convergent if and only if there is at least one $l \in \mathbb{R}$ such that $\lim a_n = l$"

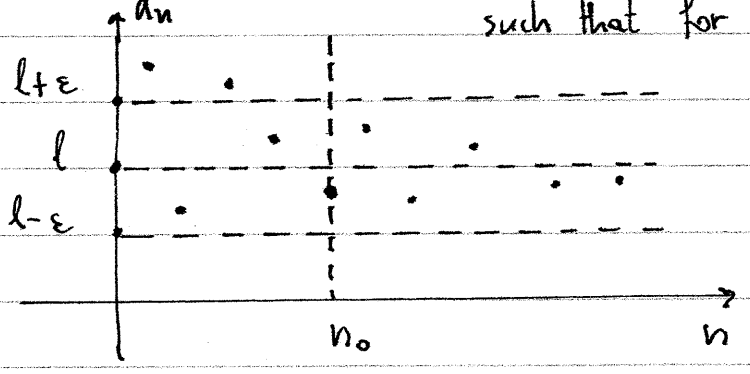
- We note that:

$$|a_n - l| < \varepsilon \Leftrightarrow -\varepsilon < a_n - l < \varepsilon \Leftrightarrow$$
$$\Leftrightarrow l - \varepsilon < a_n < l + \varepsilon$$

and therefore the inequality $|a_n - l| < \varepsilon$ means that the distance between a_n and l is less than ε.

- Intuitively, this definition means that the sequence a_n approaches l for large n. More precisely: for any maximum allowed distance $\varepsilon > 0$ (which we can make as small as we want) there is some integer $n_0 \in \mathbb{N}^*$ such that all a_n <u>after</u> a_{n_0} (with $n > n_0$) have distance from l that is less than ε. The key is that some $n_0 \in \mathbb{N}$ can be found for any choice made for $\varepsilon > 0$ regardless of how small ε is chosen.

- <u>Geometric interpretation:</u> For any $\varepsilon > 0$, there is some $n_0 \in \mathbb{N}^*$ such that for $n > n_0$, the sequence is thereafter restricted in the strip between $l - \varepsilon$ and $l + \varepsilon$.



Properties of convergent sequences

1) **Uniqueness:** Let a_n be a sequence and $l_1, l_2 \in \mathbb{R}$.

Then:

$$\begin{cases} \lim a_n = l_1 \\ \lim a_n = l_2 \end{cases} \Rightarrow l_1 = l_2$$

2) **Convergence and operations**

Let a_n, b_n be convergent sequences and let $\lambda \in \mathbb{R}$, $k \in \mathbb{N}^*$

Then:

$$\lim (a_n + b_n) = \lim a_n + \lim b_n$$

$$\lim (a_n b_n) = (\lim a_n)(\lim b_n)$$

$$\lim (\lambda a_n) = \lambda \lim a_n$$

$$\lim \left(\frac{a_n}{b_n} \right) = \frac{\lim a_n}{\lim b_n}$$

$$(\forall n \in \mathbb{N}^* : a_n \geq 0) \Rightarrow \lim \sqrt[k]{a_n} = \sqrt[k]{\lim a_n}$$

3) **Sequence and function limit**

Let a_n be a sequence, $f : [0, +\infty) \to \mathbb{R}$ be a function and let $l \in \mathbb{R}$. Then

$$\begin{cases} \forall n \in \mathbb{N}^* : a_n = f(n) \\ \lim_{x \to +\infty} f(x) = l \end{cases} \Rightarrow \lim a_n = l$$

Useful applications

The following statements are good examples and can also be used to do other exercises.

1) $\boxed{\forall a \in (0,+\infty): \lim \sqrt[n]{a} = 1}$

Proof

We note that

$$a_n \equiv \sqrt[n]{a} = a^{1/n} = \exp\left((1/n)\ln a\right) = \exp\left(\frac{\ln a}{n}\right), \forall n \in \mathbb{N}^*.$$

and since

$$\lim_{x\to+\infty} \frac{\ln a}{x} = \ln a \cdot \lim_{x\to+\infty} \frac{1}{x} = 0 \cdot \ln a = 0 \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} \exp\left(\frac{\ln a}{x}\right) = \exp(0) = 1 \Rightarrow \lim \sqrt[n]{a} = 1.$$

2) $\boxed{\lim \sqrt[n]{n} = 1}$

Proof

We note that:

$$\sqrt[n]{n} = n^{1/n} = \exp\left((1/n)\ln(n)\right) = \exp\left(\frac{\ln(n)}{n}\right), \forall n \in \mathbb{N}^*$$

and thus, since

$$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{(\ln x)'}{(x)'} = \lim_{x\to+\infty} \frac{1/x}{1} = \lim_{x\to+\infty} \frac{1}{x} = 0$$

$$\Rightarrow \lim_{x\to+\infty} \exp\left(\frac{\ln x}{x}\right) = \exp(0) = 1 \Rightarrow \lim \sqrt[n]{n} = 1.$$

3) $\boxed{\lim\left(1+\frac{a}{n}\right)^n = e^a \ , \forall a \in \mathbb{R}}$

Proof

We note that

$$\left(1+\frac{a}{n}\right)^n = \exp\left[n \ln\left(1+\frac{a}{n}\right)\right] \ , \forall n \in \mathbb{N}^*$$

and since

$$\lim_{x\to+\infty}\left[x \ln\left(1+\frac{a}{x}\right)\right] = \lim_{x\to+\infty} \frac{\ln(1+a/x)}{1/x} \overset{0/0}{=}$$

$$= \lim_{x\to+\infty} \frac{[\ln(1+a/x)]'}{(1/x)'} = \lim_{x\to+\infty} \frac{(1+a/x)^{-1}(1+a/x)'}{-1/x^2}$$

$$= \lim_{x\to+\infty} \frac{(-x^2)(-a/x^2)}{1+a/x} = \lim_{x\to+\infty} \frac{a}{1+a/x} =$$

$$= \frac{a}{1+0} = a \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} \exp\left(x \ln\left(1+\frac{a}{x}\right)\right) = \exp(a) = e^a$$

$$\Rightarrow \lim(1+a/n)^n = e^a .$$

4) $\boxed{0<a<1 \Rightarrow \lim a^n = 0}$

Proof

Define $\forall x \in \mathbb{R} : f(x) = a^x = \exp(x \ln a)$

Since $0<a<1 \Rightarrow \ln a < 0$, and therefore:

$$\lim_{x\to+\infty}(x \ln a) = -\infty \Rightarrow \lim_{x\to+\infty} a^x = \lim_{x\to+\infty} \exp(x \ln a) = 0 \Rightarrow$$

$$\Rightarrow \lim a^n = 0.$$

EXAMPLES

a) Evaluate the limit of $a_n = \frac{3n^2-4n+1}{5-2n^2}$, $\forall n \in \mathbb{N}^*$

Solution

Since

$$\lim_{x\to+\infty} \frac{3x^2-4x+1}{5-2x^2} = \lim_{x\to+\infty} \frac{3x^2}{-2x^2} = \frac{3}{-2} = \frac{-3}{2} \Rightarrow$$

$$\Rightarrow \lim a_n = \lim_{n\in\mathbb{N}^*} \frac{3n^2-4n+1}{5-2n^2} = \frac{-3}{2}$$

b) Evaluate the limit of $a_n = \frac{3^n+4^{n+1}}{3^n-4^n}$, $\forall n \in \mathbb{N}^*$

Solution

Since:

$$\forall n\in\mathbb{N}^*: a_n = \frac{3^n+4^{n+1}}{3^n-4^n} = \frac{3^n+4\cdot 4^n}{3^n-4^n} = \frac{4^n\left[(3/4)^n+4\right]}{4^n\left[(3/4)^n-1\right]}$$

$$= \frac{(3/4)^n+4}{(3/4)^n-1}$$

$$\Rightarrow \lim a_n = \lim \frac{(3/4)^n+4}{(3/4)^n-1} = \frac{4+\lim (3/4)^n}{-1+\lim (3/4)^n} = \frac{4+0}{-1+0} = -4.$$

↳ The method above is to factor out the n^{th} power of the largest number from both numerator and denominator, and use the result

$$0<a<1 \Rightarrow \lim_{n\in\mathbb{N}^*} a^n = 0.$$

c) Evaluate the limit $a_n = \sqrt{n^2+3n} - \sqrt{n^2-3n}$, $\forall n \in \mathbb{N}^* - \{1,2\}$

Solution

Since

$$\lim_{x\to+\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-3x}\right) = \lim_{x\to+\infty} \frac{\left(\sqrt{x^2+3x}\right)^2 - \left(\sqrt{x^2-3x}\right)^2}{\sqrt{x^2+3x} + \sqrt{x^2-3x}}$$

$$= \lim_{x\to+\infty} \frac{(x^2+3x)-(x^2-3x)}{\sqrt{x^2+3x} + \sqrt{x^2-3x}} = \lim_{x\to+\infty} \frac{6x}{\sqrt{x^2+3x} + \sqrt{x^2-3x}}$$

$$= \lim_{x\to+\infty} \frac{6x}{|x|\sqrt{1+3/x} + |x|\sqrt{1-3/x}} =$$

$$= \lim_{x\to+\infty} \frac{6}{\sqrt{1+3/x} + \sqrt{1-3/x}} =$$

$$= \frac{6}{\sqrt{1+0} + \sqrt{1-0}} = \frac{6}{1+1} = 3 \Rightarrow$$

$$\Rightarrow \lim a_n = 3.$$

EXERCISES

① Evaluate the limit of the following sequences

a) $a_n = \dfrac{2n^3+4n^2-2}{6n+3n^3-5}$

b) $a_n = \dfrac{n^2+2n+3}{3n^3+n^2-1}$

c) $a_n = \dfrac{n-3}{n^3+n+4}$

d) $a_n = \dfrac{2n^2-5n+3}{1-5n^2}$

e) $a_n = \dfrac{2^n+5^n}{4^n+7^n}$

f) $a_n = \dfrac{2\cdot 8^n - 5^{2n}}{4+6^{2n+1}}$

g) $a_n = \dfrac{5+3^n+5^{n+1}}{7+2^n+5^{n+4}}$

h) $a_n = \dfrac{7^n-2\cdot 5^n+2}{3^n+7^{n+1}-3}$

i) $a_n = \dfrac{3\cdot 4^n-5^{2n}}{7+6^{2n+1}}$

j) $a_n = \dfrac{2\cdot 5^n-3^{2n}}{6+4^{2n+1}}$

k) $a_n = (\sqrt{n+4}-\sqrt{n+3})\sqrt{n+2}$

l) $a_n = \sqrt{n+\sqrt{n}}-\sqrt{n-\sqrt{n}}$

m) $a_n = \sqrt{n^2+5x+1}-\sqrt{n^2+2n+3}$

o) $a_n = \sqrt{4n^2+n}-2n$

② Show that $\lim_{n\in\mathbb{N}}\left[\sqrt{(n+a)(n+b)}-n\right] = \dfrac{a+b}{2}$

Divergent sequences

- Let a_n be a sequence. We say that.

$$\boxed{a_n \text{ divergent} \iff a_n \text{ not convergent}}$$

▸ Preliminaries: If p is a proposition (i.e. a statement that is true or false) then the negation $\overline{p}$ is the statement that "p is false".

e.g. $\overline{x=3} \iff x \neq 3$

$\overline{x \geq 3} \iff x < 3$

$\overline{x \in 3} \iff x > 3$

To negate quantified statements we use the following negation rules:

$$\boxed{\begin{aligned} \overline{\forall x \in A : p(x)} &\iff \exists x \in A : \overline{p(x)} \\ \overline{\exists x \in A : p(x)} &\iff \forall x \in A : \overline{p(x)} \end{aligned}}$$

- It follows that:

$$\begin{aligned} a_n \text{ divergent} &\iff \overline{a_n \text{ convergent}} \\ &\iff \overline{\exists l \in \mathbb{R} : \lim a_n = l} \\ &\iff \forall l \in \mathbb{R} : \lim a_n = l \\ &\iff \forall l \in \mathbb{R} : \lim a_n \neq l \end{aligned}$$

and

$$\lim a_n \neq \ell \Leftrightarrow \overline{\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon}$$
$$\Leftrightarrow \exists \varepsilon > 0 : \overline{\exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon}$$
$$\Leftrightarrow \exists \varepsilon > 0 : \forall n_0 \in \mathbb{N}^* : \overline{\forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon}$$
$$\Leftrightarrow \exists \varepsilon > 0 : \forall n_0 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_0] : \overline{|a_n - \ell| < \varepsilon}$$
$$\Leftrightarrow \exists \varepsilon > 0 : \forall n_0 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_0] : |a_n - \ell| \geqslant \varepsilon$$

Combining the above statements gives:

$$\boxed{\begin{array}{l} a_n \text{ divergent} \Leftrightarrow \\ \forall \ell \in \mathbb{R} : \exists \varepsilon > 0 : \forall n_0 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_0] : |a_n - \ell| \geqslant \varepsilon \end{array}}$$

which translates:
"a_n is divergent if and only if for all $\ell \in \mathbb{R}$, there is at least one $\varepsilon > 0$ such that for all $n_0 \in \mathbb{N}^*$, there is at least one $n \in \mathbb{N}^* - [n_0]$ such that $|a_n - \ell| \geqslant \varepsilon$".

↕ Limits of subsequences

The most useful result involving subsequences is the following:

- $\boxed{\lim a_n = \ell \Leftrightarrow (\lim a_{2n} = \ell \wedge \lim a_{2n+1} = \ell)}$

An immediate consequence of this statement is that:

$$\begin{cases} \lim a_{2n} = l_1 \\ \lim a_{2n+1} = l_2 \Rightarrow a_n \text{ divergent} \\ l_1 \neq l_2 \end{cases}$$

Proof

Assume that $\lim a_{2n} = l_1 \wedge \lim a_{2n+1} = l_2 \wedge l_1 \neq l_2$.
To show that a_n divergent we assume that a_n is convergent. Then:

a_n convergent $\Rightarrow \exists l \in \mathbb{R} : \lim a_n = l$.

and it follows for a chosen $l \in \mathbb{R}$ that

$$\lim a_n = l \Rightarrow \begin{cases} \lim a_{2n} = l \\ \lim a_{2n+1} = l \end{cases} \Rightarrow \begin{cases} l_1 = l \\ l_2 = l \end{cases} \Rightarrow l_1 = l_2$$

which is a contradiction since $l_1 \neq l_2$. It follows that a_n is divergent. $\square$

- These results can be generalized to a great extent, but we will not be needing the most general formulation. The following generalization is sufficient for our needs:

▶ $$(\forall p \in [k] : \lim a_{kn+p-1} = l) \Leftrightarrow \lim a_n = l$$

▶ $$\begin{cases} \exists p, q \in [k] : (\lim a_{kn+p-1} = l_1 \wedge \lim a_{kn+q-1} = l_2) \Rightarrow \\ l_1 \neq l_2 \end{cases}$$
$$\Rightarrow a_n \text{ divergent}$$

- For $k=2$, the general statements simplify back to the original statements involving $\lim a_{2n}$ and $\lim a_{2n+1}$.
- For $k=3$, we get:

$\lim a_n = l \Leftrightarrow (\lim a_{3n} = l \wedge \lim a_{3n+1} = l \wedge \lim a_{3n+2} = l)$

and also:

$\left.\begin{array}{l} \lim a_{3n} = l_1 \\ \lim a_{3n+1} = l_2 \\ l_1 \neq l_2 \end{array}\right\} \Rightarrow a_n$ divergent

$\left.\begin{array}{l} \lim a_{3n} = l_1 \\ \lim a_{3n+2} = l_2 \\ l_1 \neq l_2 \end{array}\right\} \Rightarrow a_n$ divergent

$\left.\begin{array}{l} \lim a_{3n+1} = l_1 \\ \lim a_{3n+2} = l_2 \\ l_1 \neq l_2 \end{array}\right\} \Rightarrow a_n$ divergent.

(3 combinations)

and so on for larger k.

EXAMPLES

a) Show that $a_n = \sin\left(\frac{n\pi}{2}\right)$ is divergent.

Solution

We note that

$\forall n \in \mathbb{N}^*: \; a_{4n} = \sin\left(\frac{(4n)\pi}{2}\right) = \sin(2n\pi) = \sin(0) = 0$

$\Rightarrow \lim a_{4n} = 0 \qquad (1)$

and

$\forall n \in \mathbb{N}^*: \; a_{4n+1} = \sin\left(\frac{(4n+1)\pi}{2}\right) = \sin\left(\frac{4n\pi}{2} + \frac{\pi}{2}\right) =$

$= \sin(2n\pi + \pi/2) = \sin(\pi/2) = 1$

$\Rightarrow \lim a_{4n+1} = 1 \qquad (2)$

From Eq. (1) and Eq. (2) it follows that a_n divergent.

b) Show that $a_n = (-1)^n \frac{3n+5}{2n+1}$ is divergent.

Solution

We note that

$a_{2n} = (-1)^{2n} \frac{3(2n)+5}{2(2n)+1} = \frac{6n+5}{4n+1}, \; \forall n \in \mathbb{N}^*$

$a_{2n+1} = (-1)^{2n+1} \frac{3(2n+1)+5}{2(2n+1)+1} = (-1)\frac{6n+3+5}{4n+2+1} = \frac{-(6n+8)}{4n+3}, \; \forall n \in \mathbb{N}^*$

and we now argue that:

$$\lim_{x\to+\infty}\frac{6x+5}{4x+1}=\lim_{x\to+\infty}\frac{6x}{4x}=\frac{6}{4}=\frac{3}{2}\Rightarrow$$

$$\Rightarrow \lim a_{2n}=\lim\frac{6n+5}{4n+1}=\frac{3}{2} \qquad (1)$$

and

$$\lim_{x\to+\infty}\frac{-(6x+8)}{4x+3}=\lim_{x\to+\infty}\frac{-6x}{4x}=\frac{-6}{4}=\frac{-3}{2}\Rightarrow$$

$$\Rightarrow \lim a_{2n+1}=\lim\frac{-(6n+1)}{4n+3}=\frac{-3}{2} \qquad (2)$$

From Eq.(1) and Eq.(2) it follows that a_n is divergent.

c) Show that for $a_n=(-1)^n\frac{3n}{n^2+1}$, $\lim a_n=0$.

Solution

We note that

$$a_{2n}=(-1)^{2n}\frac{3(2n)}{(2n)^2+1}=\frac{6n}{4n^2+1},\ \forall n\in\mathbb{N}^*$$

$$a_{2n+1}=(-1)^{2n+1}\frac{3(2n+1)}{(2n+1)^2+1}=(-1)\frac{6n+3}{4n^2+4n+1+1}=$$

$$=\frac{-(6n+3)}{4n^2+4n+2},\ \forall n\in\mathbb{N}^*$$

and also that

$$\lim_{x\to+\infty}\frac{6x}{4x^2+1}=\lim_{x\to+\infty}\frac{6x}{4x^2}=\lim_{x\to+\infty}\frac{6}{4x}=0\Rightarrow$$

$$\Rightarrow \lim a_{2n}=\lim\frac{6n}{4n^2+1}=0 \qquad (1)$$

and

$$\lim_{x\to+\infty} \frac{-(6x+3)}{4x^2+4x+2} = \lim_{x\to+\infty} \frac{-6x}{4x^2} = \lim_{x\to+\infty} \frac{-6}{4x} = 0 \Rightarrow$$

$$\Rightarrow \lim a_{2n+1} = \lim \frac{-(6n+3)}{4n^2+4n+2} = 0 \qquad (2)$$

From Eq. (1) and Eq. (2): $\lim a_n = 0$.

↳ You should know how to state the definitions for limit statements and their negations, as follows:

d) State the definition for $\lim \alpha_n = 3$

Solution

$\lim \alpha_n = 3 \Leftrightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - 3| < \varepsilon$

or

$\lim a_n = 3$ if and only if for all $\varepsilon > 0$, there is some $n_0 \in \mathbb{N}^*$ such that for all natural numbers n with $n > n_0$ we have $|a_n - 3| < \varepsilon$.

e) State the definition for $\lim \alpha_n \neq 5$

Solution

$\lim \alpha_n \neq 5 \Leftrightarrow \exists \varepsilon > 0 : \forall n_0 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_0] : |\alpha_n - 5| \geq \varepsilon$

or

$\lim a_n \neq 5$ if and only if there is some $\varepsilon > 0$ such that for all $n_0 \in \mathbb{N}^*$ there is some natural number n with $n > n_0$ such that $|a_n - 5| \geq \varepsilon$.

EXERCISES

(3) Show that the following sequences are divergent.

a) $a_n = \dfrac{(-1)^n + (-1)}{2}$

b) $a_n = (-1)^n \dfrac{n+2}{3n}$

c) $a_n = \cos\left(\dfrac{n\pi}{6}\right)$

d) $a_n = \dfrac{2(-2)^n + 2^n}{(-2)^n - 3\cdot 2^{n-1}}$

e) $a_n = \dfrac{n^2 + (-1)^n n^2}{(n+2)^2}$

f) $a_n = \dfrac{(-1)^n n^2 + 3n - 1}{3n^2 + (-1)^n n + 1}$

(4) Determine whether the following sequences are convergent or divergent, and find the limit, if the sequence is convergent

a) $a_n = \dfrac{(-1)^n n \cos(n\pi)}{n+2}$

b) $a_n = (-1)^n \sin\left(\dfrac{n\pi}{4}\right)$

c) $a_n = \sqrt{3n^2 + (-1)^n n} - \sqrt{3n^2 - 2n}$

d) $a_n = (-1)^n \sin\left(\dfrac{n}{2} + (-1)^n n\pi\right)$

e) $a_n = \dfrac{2n[(-1)^n n^2 + 1]}{(-1)^n n^3 + 2}$

f) $a_n = \sin\left(n\pi + \dfrac{2n}{3}\right) + \sin\left(n\pi + \dfrac{4n}{3}\right)$

Bounded sequences

Let a_n be a sequence. We define:

a_n upper bounded $\Leftrightarrow \exists c \in \mathbb{R} : \forall n \in \mathbb{N}^* : a_n \leq c$

a_n lower bounded $\Leftrightarrow \exists c \in \mathbb{R} : \forall n \in \mathbb{N}^* : a_n \geq c$

a_n bounded $\Leftrightarrow$ { a_n upper bounded; a_n lower bounded

a_n absolutely bounded $\Leftrightarrow \exists c \in (0, +\infty) : \forall n \in \mathbb{N}^* : |a_n| \leq c$

Properties of bounded sequences

The following properties can be derived from the definitions above, for a given sequence a_n.

1) Equivalence of absolute bounded and bounded

a_n bounded $\Leftrightarrow$ a_n absolutely bounded

2) Convergence and bounded property

a_n convergent $\Rightarrow$ a_n bounded

↳ In general, given a statement $p \Rightarrow q$, the contrapositive statement $\bar{q} \Rightarrow \bar{p}$ is also true. Consequently it follows that

a_n not bounded $\Rightarrow$ a_n divergent

3) Zero-bounded theorem for sequences

$\left.\begin{array}{l} \lim a_n = 0 \\ b_n \text{ bounded} \end{array}\right\} \Rightarrow \lim (a_n b_n) = 0$

4) Unbounded sequences

$\left.\begin{array}{l} \forall n \in \mathbb{N}^* : a_n \geq f(n) \\ \lim\limits_{x \to +\infty} f(x) = +\infty \end{array}\right\} \Rightarrow a_n \text{ not bounded}$

$\left.\begin{array}{l} \forall n \in \mathbb{N}^* : a_n \leq f(n) \\ \lim\limits_{x \to +\infty} f(x) = -\infty \end{array}\right\} \Rightarrow a_n \text{ not bounded}$

Methodology

- To establish that a sequence is absolutely bounded, we use the following properties of absolute values

$\forall a,b\in\mathbb{R}: |a+b| \leq |a|+|b|$ $\quad$ $\forall a,b\in\mathbb{R}: |ab| = |a||b|$

$\forall a,b\in\mathbb{R}: |a-b| \leq |a|+|b|$ $\quad$ $\forall a\in\mathbb{R}: \forall b\in\mathbb{R}-\{0\}: \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

as well as the following inequalities

$\forall x\in\mathbb{R}: |\sin x| \leq 1$ $\quad$ $\forall x\in\mathbb{R}: |\operatorname{Arctan}(x)| < \pi/2$

$\forall x\in\mathbb{R}: |\cos x| \leq 1$ $\quad$ $\forall x\in[-1,1]: |\operatorname{Arcsin}(x)| \leq \pi/2$

and the identities

$\forall x\in\mathbb{R}: \operatorname{Arctan}(x) + \operatorname{Arccot}(x) = \pi/2$

$\forall x\in[-1,1]: \operatorname{Arcsin}(x) + \operatorname{Arccos}(x) = \pi/2$

- To show any statement of the form $\forall x\in A: p(x)$ we write:

> Let $x\in A$ be given.
> [Prove $p(x)$]
> It follows that $\forall x\in A: p(x)$

EXAMPLES

a) Show that $a_n = \dfrac{\sin(3\pi n) + \cos(\pi n)}{n^2+1}$ is convergent.

Solution

Define $\forall n \in \mathbb{N}^*$: $b_n = 1/(n^2+1)$

and $\forall n \in \mathbb{N}^*$: $c_n = \sin(3\pi n) + \cos(\pi n)$

and note that $\forall n \in \mathbb{N}^*$: $a_n = b_n c_n$.

We note that:

$$\lim_{x\to+\infty} \frac{1}{x^2+1} = \lim_{x\to+\infty} \frac{1}{x^2} = 0 \Rightarrow \lim b_n = \lim \frac{1}{n^2+1} = 0 \quad (1)$$

Let $n \in \mathbb{N}^*$ be given. Then

$$|c_n| = |\sin(3\pi n) + \cos(\pi n)| \leq |\sin(3n\pi)| + |\cos(\pi n)|$$
$$\leq 1 + 1 = 2$$

It follows that

$(\forall n \in \mathbb{N}^*: |c_n| \leq 2) \Rightarrow c_n$ absolutely bounded

$\Rightarrow c_n$ bounded (2).

From Eq. (1) and Eq. (2):

$$\begin{cases} \lim b_n = 0 \\ c_n \text{ bounded} \end{cases} \Rightarrow \lim(b_n c_n) = 0 \Rightarrow \lim a_n = 0$$

$\Rightarrow a_n$ convergent

b) Show that $a_n = (-1)^n \operatorname{Arccos}((-1)^n/3)\sin(\pi/n)$ is convergent

Solution

We define:

$\forall n \in \mathbb{N}^*: b_n = (-1)^n \operatorname{Arccos}((-1)^n/3)$

$\forall n \in \mathbb{N}^*: c_n = \sin(\pi/n)$

Let $n \in \mathbb{N}^*$ be given. Then:

$$|b_n| = |(-1)^n \operatorname{Arccos}((-1)^n/3)| = |(-1)^n||\operatorname{Arccos}((-1)^n/3)|$$
$$= |\operatorname{Arccos}((-1)^n/3)| = |\pi/2 - \operatorname{Arcsin}((-1)^n/3)|$$
$$\leq |\pi/2| + |\operatorname{Arcsin}((-1)^n/3)| \leq \pi/2 + \pi/2 = \pi$$

and therefore:

$(\forall n \in \mathbb{N}^*: |b_n| \leq \pi) \Rightarrow b_n$ absolutely bounded

$\Rightarrow b_n$ bounded (1)

We also note that:

$$\lim_{x \to +\infty} \frac{\pi}{x} = 0 \Rightarrow \lim_{x \to +\infty} \sin\left(\frac{\pi}{x}\right) = \sin(0) = 0 \Rightarrow$$

$$\Rightarrow \lim c_n = \lim \sin(\pi/n) = 0 \quad (2)$$

From Eq.(2) and Eq.(3):

$$\begin{cases} \lim c_n = 0 \\ b_n \text{ bounded} \end{cases} \Rightarrow \lim(b_n c_n) = 0 \Rightarrow \lim a_n = 0$$

$\Rightarrow a_n$ convergent.

↳ To show that a sequence is bounded
1) We show it is absolutely bounded ; OR
2) We show it is convergent.

c) Show that the sequence

$$a_n = \frac{n\cos(n\pi/4) - n^2\sin(n\pi)}{n^3+1}, \quad \forall n \in \mathbb{N}^*$$

is bounded.

Solution

Let $n \in \mathbb{N}^*$ be given. Then:

$$|a_n| = \left|\frac{n\cos(n\pi/4) - n^2\sin(n\pi)}{n^3+1}\right| = \frac{|n\cos(n\pi/4) - n^2\sin(n\pi)|}{|n^3+1|}$$

$$\leq \frac{|n\cos(n\pi/4)| + |n^2\sin(n\pi)|}{n^3+1} = \frac{n|\cos(n\pi/4)| + n^2|\sin(n\pi)|}{n^3+1}$$

$$\leq \frac{n+n^2}{n^3+1} \leq \frac{n^2+n^2}{n^3+1} = \frac{2n^2}{n^3+1} \leq \frac{2n^2}{n^3} =$$

$$= \frac{2}{n} \leq 2$$

and it follows that

$(\forall n \in \mathbb{N}^* : |a_n| \leq 2) \Rightarrow a_n$ absolutely bounded

$\Rightarrow a_n$ bounded. □

↳ Note that to find an upper bound for a fraction with positive numerator and denominator, we can increase the fraction by increasing the numerator or by decreasing the denominator, using $n \geq 1$.

d) Show that the sequence

$$\forall n \in \mathbb{N}^*: \; a_n = \frac{n^2 - n + 1}{n^2 + n + 1}$$

is bounded.

Solution

1st method

Since

$$\lim_{x \to +\infty} \frac{x^2 - x + 1}{x^2 + x + 1} = \lim_{x \to +\infty} \frac{x^2}{x^2} = 1 \Rightarrow$$

$\Rightarrow \lim a_n = 1 \Rightarrow a_n$ convergence $\Rightarrow a_n$ bounded.

2nd method

Let $n \in \mathbb{N}^*$ be given. Then:

$$|a_n| = \left| \frac{n^2 - n + 1}{n^2 + n + 1} \right| = \frac{|n^2 - n + 1|}{|n^2 + n + 1|} = \frac{|n^2 - n + 1|}{n^2 + n + 1} \leq$$

$$\leq \frac{|n^2| + |n| + 1}{n^2 + n + 1} = \frac{n^2 + n + 1}{n^2 + n + 1} = 1$$

and therefore:

$(\forall n \in \mathbb{N}^*: |a_n| \leq 1) \Rightarrow a_n$ absolutely bounded

$\Rightarrow a_n$ bounded. □

e) Show that the sequence

$$\forall n \in \mathbb{N}^*: \; a_n = \frac{n^2+1}{5n+1+n\cos(\pi n/3)}$$

is <u>not</u> bounded.

<u>Solution</u>

Let $n \in \mathbb{N}^*$ be given. Then:

$$a_n = \frac{n^2+1}{5n+1+n\cos(\pi n/3)} \geqslant \frac{n^2+1}{5n+n+n\cos(n\pi/3)} =$$

$$= \frac{n^2+1}{6n+n\cos(n\pi/3)} \geqslant \frac{n^2+1}{6n+n} = \frac{n^2+1}{7n} \geqslant$$

$$\geqslant \frac{n^2}{7n} = \frac{n}{7}$$

and therefore: $\forall n \in \mathbb{N}^*: \; a_n \geqslant n/7$ (1)

We also have: $\lim_{x \to +\infty} \frac{x}{7} = +\infty$ (2)

From Eq.(1) and Eq.(2) it follows that a_n not bounded.

EXERCISES

(5) Show that the following sequences are bounded:

a) $a_n = (-7)^n$

b) $a_n = \dfrac{n+1}{n^2+8}$

c) $a_n = \dfrac{n\cos(\pi n/3) + \sin(\pi n/4)}{n^2+3n+1}$

d) $a_n = \dfrac{1}{n}\sin\left(\dfrac{\pi n}{2}\right)$

e) $a_n = \dfrac{5\sin(3\pi n)}{4n}$

f) $a_n = \dfrac{n+5}{n+2}$

g) $a_n = \sqrt{n^2+n} - n$

h) $a_n = \dfrac{\sqrt{n+1} + \sqrt{n+2}}{\sqrt{n+3}}$

(6) Show that the following sequences are NOT bounded:

a) $a_n = \dfrac{4n^2+1}{3n}$

b) $a_n = \dfrac{n^2+2}{n+2}$

c) $a_n = -4n^2+3n+1$

d) $a_n = 2n^3 - n + 1$

e) $a_n = \dfrac{n^2}{2n + \cos^2(\pi n)}$

f) $a_n = \dfrac{2n^2+5}{3n + n\sin(\pi n)}$

(7) Use the zero-bounded theorem to evaluate the limits of the following sequences.

a) $a_n = (-1)^n \operatorname{Arccot}\left(1 + \sin(\pi^9 n)\right) \ln\left(\dfrac{n+1}{n+3}\right)$

b) $a_n = \dfrac{\sin(n\pi/3) + \cos(n\pi/4)}{n^3+3n+1}$

c) $a_n = \cos(\pi n)\left[\sqrt{n+1} - \sqrt{n}\right]$

d) $a_n = \cos(\pi n/4)\left[(-1)^n + \sin(n\pi/3)\right]\left[\sqrt{n^2+3} - \sqrt{n^2+1}\right]$

e) $a_n = (-1)^n \cos\left(\frac{\pi n}{4}\right) \sin\left(\frac{1}{n}\right)$

f) $a_n = \left[\sin\left(\frac{n\pi}{3}\right) + \cos\left(\frac{n\pi}{4}\right)\right] \ln\left(\frac{n^2+3n+1}{n^2+2n+4}\right)$

Recursive sequences and monotonicity.

- Let a_n be a sequence. Then we introduce the following definitions:

a_n increasing $\Leftrightarrow \forall n \in \mathbb{N}^*: a_{n+1} \geq a_n$
a_n decreasing $\Leftrightarrow \forall n \in \mathbb{N}^*: a_{n+1} \leq a_n$

- From the least upper bound axiom we can prove the following statements

a_n increasing a_n upper bounded	$\Rightarrow$	a_n convergent
a_n decreasing a_n lower bounded	$\Rightarrow$	a_n convergent

Methodology

These results can be used to evaluate the limit of a recursive sequence of the form

$$\begin{cases} a_0 = c \\ a_{n+1} = f(a_n) \end{cases}$$

- 1 We use the method of induction to show that a_n is increasing or decreasing.
- 2 Likewise we show that a_n is upper or lower bounded.
- 3 We conclude that a_n is convergent and define $x = \lim a_n$.
- 4 We show that $x = f(x)$ and solve for x.
  a) If there is a unique solution then that solution

is the limit

b) If there are multiple solutions then we use Bounds to eliminate all but one solution.

EXAMPLE

Evaluate the limit $\lim a_n$ of the sequence a_n given by

$$\begin{cases} a_1 = 3 \\ a_{n+1} = \sqrt{a_n + 7} \end{cases}$$

Solution

• Monotonicity.

We note that $a_2 = \sqrt{a_1 + 7} = \sqrt{3+7} = \sqrt{10} > \sqrt{9} = 3 = a_1$

$\Rightarrow a_2 > a_1$. We will thus show that a_n increasing.

Assume that for $n=k$: $a_{k+1} > a_k > 0$ (1)

We will show that for $n=k+1$: $a_{k+2} > a_{k+1} > 0$

From Eq. (1):

$$a_{k+1} > a_k > 0 \Rightarrow a_{k+1} + 7 > a_k + 7 > 7$$
$$\Rightarrow \sqrt{a_{k+1}+7} > \sqrt{a_k + 7} > \sqrt{7}$$
$$\Rightarrow a_{k+2} > a_{k+1} > 0.$$

By induction, it follows that

$(\forall n \in \mathbb{N}^*: a_{n+1} > a_n) \Rightarrow$ a_n increasing (2)

• Bounds

We will show that $\forall n \in \mathbb{N}^*: a_n < 4$.

For $n=1$: $a_1 = 3 < 4$.

Assume that for $n=k$: $\underline{0 < a_k < 4}$. (3)

For $n = k+1$ we will show that : $0 < a_{k+1} < 4$.

From Eq.(3):

$0 < a_k < 4 \Rightarrow 7 < a_k + 7 < 4+7 \rightarrow \sqrt{7} < \sqrt{a_k + 7} < \sqrt{11}$

$\Rightarrow \sqrt{7} < a_{k+1} < \sqrt{11} \Rightarrow 0 < a_{k+1} < 4$

By induction it follows that

$(\forall n \in \mathbb{N}^*: a_n < 4) \Rightarrow \underline{a_n \text{ upper bounded}}$ (4)

From Eq.(2) and Eq.(4):

$\begin{cases} a_n \text{ increasing} \\ a_n \text{ upper bounded} \end{cases} \Rightarrow a_n \text{ convergent} \Rightarrow \exists x \in \mathbb{R}: \lim x_n = x.$

Since :

$x = \lim a_{n+1} = \lim \sqrt{a_n + 7} = \sqrt{\lim a_n + 7} = \sqrt{x+7}$

we require that $\underline{x \geq 0}$ and note that

$x = \sqrt{x+7} \Leftrightarrow x^2 = x+7 \Leftrightarrow x^2 - x - 7 = 0$

$\Delta = (-1)^2 - 4 \cdot 1 \cdot (-7) = 1 + 28 = 29 \Rightarrow$

$\Rightarrow x_{1,2} = \frac{-(-1) \pm \sqrt{29}}{2 \cdot 1} = \frac{1 \pm \sqrt{29}}{2} = \begin{cases} \frac{1-\sqrt{29}}{2} & \leftarrow \text{rejected} \\ \frac{1+\sqrt{29}}{2} & \leftarrow \text{accepted} \end{cases}$

It follows that

$\lim a_n = \frac{1+\sqrt{29}}{2}$

↳ The negative solution is rejected because it violates the requirement $x \geq 0$.

EXERCISES

(8) Show that the following sequences converge and find their limit.

a) (a_n): $\begin{cases} a_1 = 1 \\ a_{n+1} = \sqrt{1 + a_n} \end{cases}$

b) (a_n): $\begin{cases} a_1 = 3 \\ a_{n+1} = \dfrac{3a_n - 4}{5} \end{cases}$

c) (a_n): $\begin{cases} a_1 = 2 \\ a_{n+1} = \dfrac{2a_n - 3}{4} \end{cases}$

d) (a_n): $\begin{cases} a_1 = 1/4 \\ a_{n+1} = \dfrac{1}{2} a_n^2 + \dfrac{1}{8} \end{cases}$

e) (a_n): $\begin{cases} a_1 = 1 \\ a_{n+1} = (1/3)\, a_n + 2 \end{cases}$

f) (a_n): $\begin{cases} a_1 = 0 \\ a_{n+1} = \dfrac{3a_n + 1}{4} \end{cases}$

g) (a_n): $\begin{cases} a_1 = 3 \\ a_{n+1} = (1/5)(a_n^2 + 4) \end{cases}$

h) (a_n): $\begin{cases} a_1 = 2 \\ a_{n+1} = \sqrt{1 + 2a_n} - 1 \end{cases}$

i) (a_n): $\begin{cases} a_1 = 2 \\ a_{n+1} = \sqrt{a_n + 6} \end{cases}$

j) (a_n): $\begin{cases} a_1 = 3 \\ a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right) \end{cases}$

k) (a_n): $\begin{cases} a_1 = 1 \\ a_{n+1} = \sqrt{3a_n + 4} \end{cases}$

Convergence and order

We consider two sequences a_n, b_n. We can show from the sequence limit definition that

(1) $\left.\begin{array}{l} \forall n \in \mathbb{N}^*: a_n \geq 0 \\ a_n \text{ convergent} \end{array}\right\} \Rightarrow \lim a_n \geq 0$

(2) $\lim a_n > 0 \Rightarrow \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: a_n > 0$

(3) $\left.\begin{array}{l} \forall n \in \mathbb{N}^*: |a_n| \leq |b_n| \\ \lim b_n = 0 \end{array}\right\} \Rightarrow \lim a_n = 0$

Note that while statement (2) holds for strict inequalities, statement (1) cannot be generalized into using only strict inequalities. For example, the sequence

$$\forall n \in \mathbb{N}^*: \; a_n = 1/n$$

satisfies the statement

$$\left\{\begin{array}{ll} \forall n \in \mathbb{N}^*: a_n > 0 & (1) \\ a_n \text{ convergent} & \end{array}\right.$$

however, it is easy to show that $\lim a_n = 0$. Therefore, given Eq. (1), the best we can do in general is to conclude that $\lim a_n \geq 0$.

Squeeze theorem

$$\left.\begin{array}{l} b_n, c_n \text{ convergent} \\ \forall n \in \mathbb{N}^*: b_n \leq a_n \leq c_n \\ \lim b_n = \lim c_n = l \end{array}\right\} \Rightarrow \lim a_n = l$$

Proof

Let $n \in \mathbb{N}^*$ be given. Then:

$b_n \leq a_n \leq c_n \Rightarrow 0 \leq a_n - b_n \leq c_n - b_n \Rightarrow |a_n - b_n| \leq |c_n - b_n|$

and it follows that

$\forall n \in \mathbb{N}^*: |a_n - b_n| \leq |c_n - b_n|$ (1)

We also note that

$\lim (c_n - b_n) = \lim c_n - \lim b_n = l - l = 0$ (2)

From Eq. (1) and Eq. (2), it follows that $\lim (a_n - b_n) = 0$ and therefore:

$\lim a_n = \lim (a_n - b_n + b_n) = \lim (a_n - b_n) + \lim b_n$

$= 0 + l = l$ □

↳ Application to limits of sequences $a_n = \sqrt[n]{f(n)}$

Recall that we have already shown that:

$$\forall a \in (0, +\infty): \lim \sqrt[n]{a} = 1$$
$$\lim \sqrt[n]{n} = 1$$

Using the squeeze theorem, we show that

$$\begin{cases} \forall n \in \mathbb{N}^*: a_n > 0 \Rightarrow \lim \sqrt[n]{a_n} = 1 \\ \lim a_n = l > 0 \end{cases}$$

Proof

$\lim a_n = \ell > 0 \Rightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon$

$\Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \ell/2$

For the existing $n_0 \in \mathbb{N}^*$, let $n \in \mathbb{N}^* - [n_0]$ be given. Then

$|a_n - \ell| < \ell/2 \Rightarrow -\ell/2 < a_n - \ell < \ell/2 \Rightarrow \ell/2 < a_n < 3\ell/2$

$\Rightarrow \sqrt[n]{\ell/2} < \sqrt[n]{a_n} < \sqrt[n]{3\ell/2}$

and therefore $\forall n \in \mathbb{N}^* - [n_0] : \sqrt[n]{\ell/2} < \sqrt[n]{a_n} < \sqrt[n]{3\ell/2}$ (1)

We also note that

$\ell/2 > 0 \Rightarrow \lim \sqrt[n]{\ell/2} = 1$ (2)

$3\ell/2 > 0 \Rightarrow \lim \sqrt[n]{3\ell/2} = 1$ (3)

From Eq.(1), Eq.(2), Eq.(3) via the squeeze theorem we conclude that

$\lim \sqrt[n]{a_n} = 1$ □

↳ D'Alembert theorem

$$\left.\begin{array}{l} \forall n \in \mathbb{N}^* : a_n > 0 \\ \lim \dfrac{a_{n+1}}{a_n} = \ell < 1 \end{array}\right\} \Rightarrow \lim a_n = 0$$

Proof

We note that

$$\lim\left(\frac{a_{n+1} - a_n}{a_n}\right) = \lim\left(\frac{a_{n+1}}{a_n} - 1\right) = \left(\lim \frac{a_{n+1}}{a_n}\right) - 1$$

$$= \ell - 1 < 1 - 1 = 0 \Rightarrow \lim\left(\frac{a_{n+1} - a_n}{a_n}\right) < 0$$

$\Rightarrow \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: \frac{a_{n+1} - a_n}{a_n} < 0$

$\Rightarrow \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: a_{n+1} - a_n < 0$

$\Rightarrow \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: a_{n+1} < a_n$

For the existing $n_0 \in \mathbb{N}^*$, it follows that

a_{n+n_0} decreasing (1)

We also note that:

$\begin{cases} a_{n+n_0} \text{ decreasing} \\ \forall n \in \mathbb{N}^*: a_n > 0 \end{cases} \Rightarrow \begin{cases} a_{n+n_0} \text{ decreasing} \\ a_n \text{ lower bounded} \end{cases} \Rightarrow$

$\Rightarrow a_{n+n_0}$ convergent

$\Rightarrow a_n$ convergent

$\Rightarrow \exists A \in \mathbb{R}: \lim a_n = A.$

Since $(\forall n \in \mathbb{N}^*: a_n > 0) \Rightarrow \lim a_n \geq 0 \Rightarrow A \geq 0$, to show that $A = 0$, we will assume that $A > 0$. Then:

$$\lim \frac{a_{n+1}}{a_n} = \frac{\lim a_{n+1}}{\lim a_n} = \frac{A}{A} = 1$$

which is a contradiction, since we have assumed that

$$\lim \frac{a_{n+1}}{a_n} < 1$$

It follows that: $A = 0 \Rightarrow \lim a_n = 0.$ □

EXAMPLES

a) Evaluate the limit of $a_n = \sqrt[n]{\frac{3n^2+2}{n^2+1}}$

Solution

We note that

$$\lim_{x\to+\infty} \frac{3x^2+2}{x^2+1} = \lim_{x\to+\infty} \frac{3x^2}{x^2} = 3 \Rightarrow \lim_{n\in\mathbb{N}^*} \frac{3n^2+2}{n^2+1} = 3 \quad (1)$$

and

$$\forall n\in\mathbb{N}^*: \frac{3n^2+2}{n^2+1} > 0 \quad (2)$$

From Eq.(1) and Eq.(2): $\lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} \sqrt[n]{\frac{3n^2+2}{n^2+1}} = 1$

b) Evaluate the limit of $a_n = \sqrt[n]{n^2+3n+1}$

Solution

We note that

$$a_n = \sqrt[n]{n^2+3n+1} = \sqrt[n]{n^2}\sqrt[n]{1+3n^{-1}+n^{-2}} =$$
$$= (\sqrt[n]{n})^2\sqrt[n]{1+3n^{-1}+n^{-2}}$$

Since:

$$\lim_{n\in\mathbb{N}^*} (1+3n^{-1}+n^{-2}) = 1+\lim_{n\in\mathbb{N}^*}(3/n)+\lim_{n\in\mathbb{N}^*}(3/n^2) =$$
$$= 1+0+0 = 1$$
$$\left.\begin{array}{l} \\ \forall n\in\mathbb{N}^*: 1+3n^{-1}+n^{-2} > 0 \end{array}\right\} \Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{1+3n^{-1}+n^{-2}} = 1$$

it follows that:

$$\lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} \left[(\sqrt[n]{n})^2 \sqrt[n]{1+3n^{-1}+n^{-2}} \right] =$$

$$= \left[\lim_{n\in\mathbb{N}^*} \sqrt[n]{n} \right]^2 \lim_{n\in\mathbb{N}^*} \sqrt[n]{1+3n^{-1}+n^{-2}}$$

$$= 1^2 1 = 1$$

c) Evaluate the limit of $a_n = \sqrt[n]{3^n+7^n}$

Solution

We note that

$$a_n = \sqrt[n]{3^n+7^n} = \sqrt[n]{7^n\left[(3/7)^n+1\right]} = \sqrt[n]{7^n}\sqrt[n]{1+(3/7)^n}$$

$$= 7\sqrt[n]{1+(3/7)^n}, \quad \forall n\in\mathbb{N}^*$$

and since

$$\lim_{n\in\mathbb{N}^*} \left[1+(3/7)^n\right] = 1 + \lim_{n\in\mathbb{N}^*} (3/7)^n = 1+0 = 1 \qquad (1)$$

$$\forall n\in\mathbb{N}^* : 1+(3/7)^n > 0 \qquad (2)$$

it follows from Eq.(1) and Eq.(2) that

$$\lim_{n\in\mathbb{N}^*} a_n = 7 \lim_{n\in\mathbb{N}^*} \sqrt[n]{1+(3/7)^n} = 7\cdot 1 = 7.$$

d) Evaluate the limit of $a_n = \dfrac{q^n}{n!}$

Solution

We note that $\forall n\in\mathbb{N}^* : a_n > 0$ (1)

and

$$\forall n\in\mathbb{N}^* : \frac{a_{n+1}}{a_n} = \frac{\left[\frac{q^{n+1}}{(n+1)!}\right]}{\frac{q^n}{n!}} = \frac{q^{n+1}\,n!}{q^n\,(n+1)!} = \frac{q\,n!}{(n+1)\,n!}$$

$$= \frac{q}{n+1} \qquad (2)$$

From Eq.(2), since

$$\lim_{x\to+\infty} \frac{2}{x+1} = \lim_{x\to+\infty} \frac{2}{x} = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \frac{a_{n+1}}{a_n} = \lim_{n\in\mathbb{N}^*} \frac{2}{n+1} = 0 < 1 \quad (3)$$

From Eq.(1) and Eq.(3): $\lim a_n = 0$

e) Evaluate the limit of $a_n = \frac{n!}{n^n}$

Solution

We note that

$\forall n \in \mathbb{N}^*: a_n > 0$ (1)

and

$$\forall n \in \mathbb{N}^*: \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{n^n (n+1)!}{n!\,(n+1)^{n+1}} =$$

$$= \frac{n^n (n+1)\, n!}{n!\,(n+1)^n (n+1)} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n =$$

$$= \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \frac{a_{n+1}}{a_n} = \lim_{n\in\mathbb{N}^*} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{\lim\limits_{n\in\mathbb{N}^*} (1+1/n)^n} = \frac{1}{e} < 1 \quad (2)$$

From Eq.(1) and Eq.(2): $\lim\limits_{n\in\mathbb{N}^*} a_n = 0$

EXERCISES

(9) Evaluate the limits of the following sequences:

a) $a_n = \sqrt[n]{n^2+1}$

b) $a_n = \sqrt[n]{5n^2+2n}$

c) $a_n = \sqrt[n]{2n^3-n+5}$

d) $a_n = \sqrt[n]{2^n+3^n+5^n}$

e) $a_n = \sqrt[n]{2^n+4^n+9^n}$

f) $a_n = \sqrt[n]{3+1/n}$

g) $a_n = \sqrt[n]{\dfrac{3n^2+n+1}{6n^2+5n-2}}$

h) $a_n = \sqrt[n]{\dfrac{7n+1}{3n+2}}$

(10) Similarly, evaluate the limits of the following sequences.

a) $a_n = \dfrac{n}{3^n}$

b) $a_n = \dfrac{n}{2^n}$

c) $a_n = \dfrac{n!}{n^n}$

d) $a_n = \dfrac{n^3}{6^n}$

e) $a_n = \dfrac{3^n}{n!}$

f) $a_n = \dfrac{n^{n+1}+2^n n!}{(2n)^n}$

g) $a_n = \dfrac{5^n+n^3}{6^n+n^2}$

h) $a_n = \dfrac{4^n n!}{(9n)^n}$

CAL 2.5: Series convergence

SERIES

Convergent series

• **Def** : Let a_n be a sequence. The **series** S_n of a_n is defined as

$$\forall n \in \mathbb{N}^*: \; S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$

↳ We oftentimes write $S_n = \sum a_n$ as an abbreviation.

• **Def** : If $S_n = \sum a_n$ is convergent, then we write:

$$\sum_{k=1}^{+\infty} a_k = \lim S_n$$

↳ **Series whose limit can be evaluated**

1) Geometric series

Since:

$$S_n = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, \quad \forall n \in \mathbb{N}^*$$

it follows that:

$$a \in (-1,1) \Rightarrow \lim S_n = \sum_{k=0}^{+\infty} a^k = \frac{1}{1-a}$$

$$a \in (-\infty,-1] \cup [1,+\infty) \Rightarrow \sum_{k=0}^{+\infty} a^k \text{ diverges}$$

2) Collapsing series

Any series of the form

$$S_n = \sum_{k=1}^{n} (a_{k+1} - a_k) = a_{n+1} - a_1$$

is called a collapsing or telescopic series, and if it converges, its limit is given by

$$\boxed{\lim \sum_{k=1}^{n} (a_{k+1} - a_k) = (\lim a_n) - a_1}$$

↳ Properties of convergent series

(1) Linearity of convergent series

Thm: If $\sum a_n, \sum b_n$ are convergent series, then

$$\boxed{\sum_{k=1}^{+\infty} (a_k + b_k) = \sum_{k=1}^{+\infty} a_k + \sum_{k=1}^{+\infty} b_k}$$

$$\boxed{\forall \lambda \in \mathbb{R}: \sum_{k=1}^{+\infty} (\lambda a_k) = \lambda \sum_{k=1}^{+\infty} a_k}$$

(2) Necessary condition for series convergence

$$\boxed{\sum_{k=1}^{+\infty} a_k \text{ converges} \Rightarrow \lim a_n = 0}$$

↳ The contrapositive statement gives a criterion for showing series divergence:

$$\boxed{\lim a_n \neq 0 \Rightarrow \sum_{k=1}^{+\infty} a_k \text{ diverges}}$$

Proof

Let $s_n = \sum_{k=1}^{n} a_k$ and assume that s_n convergent.

Then: $\exists l \in \mathbb{R} : \lim s_n = l$

and therefore:

$$\lim a_n = \lim \left[a_n + \sum_{k=1}^{n-1} a_k - \sum_{k=1}^{n-1} a_k \right] = \lim \left[\sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k \right]$$

$$= \lim (s_n - s_{n-1}) = \lim s_n - \lim s_{n-1} =$$

$$= l - l = 0 \qquad \square$$

EXAMPLES

Evaluate the following series, if they converge, or show that they diverge.

a) $\sum_{k=1}^{+\infty} \frac{1}{(k+2)(k+3)}$

Solution

We note that

$$\forall n \in \mathbb{N}^*: a_n = \frac{1}{(n+2)(n+3)} = \frac{A}{n+2} + \frac{B}{n+3}$$

with

$$A = \left.\frac{1}{x+3}\right|_{x=-2} = \frac{1}{(-2)+3} = \frac{1}{1} = 1$$

$$B = \left.\frac{1}{x+2}\right|_{x=-3} = \frac{1}{(-3)+2} = \frac{1}{(-1)} = -1$$

and therefore:

$$\sum_{k=1}^{+\infty} \frac{1}{(k+2)(k+3)} = \sum_{k=1}^{+\infty} \left[\frac{1}{k+2} - \frac{1}{k+3}\right] =$$

$$= \lim_{n\in\mathbb{N}} \sum_{k=1}^{n} \left[\frac{1}{k+2} - \frac{1}{k+3}\right] = \lim_{n\in\mathbb{N}} \left[\frac{1}{3} - \frac{1}{n+3}\right]$$

$$= \frac{1}{3} - \lim \frac{1}{n+3} = \frac{1}{3} - 0 = \frac{1}{3}$$

↳ This is an example of a telescopic series.

B) $\sum_{k=2}^{+\infty} \frac{3^k - 2^k}{7^k}$

Solution

We note that

$\forall n \in \mathbb{N}^*: \; S_n = \sum_{k=2}^{n} \frac{3^k - 2^k}{7^k} = \sum_{k=0}^{n} \frac{3^k - 2^k}{7^k} - \sum_{k=0}^{1} \frac{3^k - 2^k}{7^k} =$

$= \sum_{k=0}^{n} \left[\left(\frac{3}{7}\right)^k - \left(\frac{2}{7}\right)^k \right] - \frac{3-2}{7} - \frac{1-1}{1} =$

$= \frac{1-(3/7)^{n+1}}{1-(3/7)} - \frac{1-(2/7)^{n+1}}{1-(2/7)} - \frac{1}{7} \Rightarrow$

$\Rightarrow \sum_{k=2}^{+\infty} \frac{3^k - 2^k}{7^k} = \lim S_n = \lim \left[\frac{1-(3/7)^{n+1}}{1-(3/7)} - \frac{1-(2/7)^{n+1}}{1-(2/7)} - \frac{1}{7} \right] =$

$= \frac{1 - \lim (3/7)^{n+1}}{1-(3/7)} - \frac{1 - \lim (2/7)^{n+1}}{1-(2/7)} - \frac{1}{7} =$

$= \frac{1}{1-(3/7)} - \frac{1}{1-(2/7)} - \frac{1}{7} =$

$= \frac{7}{7-3} - \frac{7}{7-2} - \frac{1}{7} = \frac{7}{4} - \frac{7}{5} - \frac{1}{7} =$

$= \frac{7 \cdot 5 \cdot 7 - 7 \cdot 4 \cdot 7 - 4 \cdot 5}{4 \cdot 5 \cdot 7} = \frac{245 - 196 - 20}{140}$

$= \frac{29}{140}$

c) $\sum_{n=1}^{+\infty} \frac{n^3}{3n^3+2n^2+1}$

Solution

Since:

$$\lim_{x\to+\infty} \frac{x^3}{3x^3+2x^2+1} = \lim_{x\to+\infty} \frac{x^3}{3x^3} = \frac{1}{3} \Rightarrow$$

$$\Rightarrow \lim \frac{n^3}{3n^3+2n^2+1} = \frac{1}{3} \neq 0 \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{n^3}{3n^3+2n^2+1} \text{ diverges.}$$

Convergence of positive series

- A positive series $S_n = \sum_{k=1}^{n} a_k$ is any series that satisfies:
  $\forall k \in \mathbb{N}^*: a_k \geq 0$.
- The convergence of positive series can be determined by the following tests:
  1) Integral test
  2) Comparison test
  3) Limit comparison test
  4) Ratio test
  5) Root test

(1) → Integral test

Thm: Let $f: [1, +\infty) \to \mathbb{R}$ be a function such that

$$\begin{cases} f \text{ continuous on } [1, +\infty) \\ f \searrow [1, +\infty) \wedge (\forall x \in [1, +\infty): f(x) > 0) \\ \forall n \in \mathbb{N}^*: a_n = f(n) \end{cases}$$

Then:

a) $\sum_{k=1}^{+\infty} a_k$ converges $\iff \int_1^{+\infty} f(x)\,dx$ converges

b) $\sum_{k=1}^{+\infty} a_k$ diverges $\iff \int_1^{+\infty} f(x)\,dx$ diverges.

Proof

Since,

$$f \searrow [1,+\infty) \Rightarrow \forall k \in \mathbb{N}^* : \forall x \in [k,k+1] : f(k) \geq f(x) \geq f(k+1)$$
$$\Rightarrow \forall k \in \mathbb{N}^* : \left(\min_{x \in [k,k+1]} f(x) = f(k+1) \wedge \max_{x \in [k,k+1]} f(x) = f(k)\right)$$

it follows that the Riemann sums of f are given by

$$L_n(f|_{1,n}) = \sum_{k=1}^{n-1} \left[\min_{x \in [k,k+1]} f(x)\right]((k+1)-k) =$$
$$= \sum_{k=1}^{n-1} f(k+1) = \sum_{k=2}^{n} f(k) = \sum_{k=2}^{n} a_k$$

and

$$U_n(f|_{1,n}) = \sum_{k=1}^{n-1} \left[\max_{x \in [k,k+1]} f(x)\right]((k+1)-k) =$$
$$= \sum_{k=1}^{n-1} f(k) = \sum_{k=1}^{n-1} a_k.$$

Define $\forall n \in \mathbb{N}^* : I_n = \int_1^n f(t)\,dt$

Then: $\forall n \in \mathbb{N}^* : L_n(f|_{1,n}) \leq I_n \leq U_n(f|_{1,n})$.

($\Rightarrow$): Assume that $\sum_{k=1}^{+\infty} a_k$ converges.

We define

$$\forall x \in [1,+\infty) : F(x) = \int_1^x f(t)\,dt$$

$$\rightarrow \forall x \in [1,+\infty) : F'(x) = \frac{d}{dx}\int_1^x f(t)\,dt = f(x) > 0$$

$\Rightarrow F \nearrow [1, +\infty)$ (1)

▸ We will show that I_n is increasing and upper bounded.

Let $n \in \mathbb{N}^*$ be given. Then:

$$I_n = \int_1^n f(t)\,dt = F(n) < F(n+1) = \int_1^{n+1} f(t)\,dt = I_{n+1} \Rightarrow I_n < I_{n+1}.$$

and therefore:

$(\forall n \in \mathbb{N}^* : I_n < I_{n+1}) \Rightarrow I_n$ increasing. (2)

Let $n \in \mathbb{N}^*$ be given. Then:

$$I_n \leq U_n(f|1,n) = \sum_{k=1}^{n-1} a_k \leq \sum_{k=1}^{+\infty} a_k$$

and therefore:

$$(\forall n \in \mathbb{N}^* : I_n \leq \sum_{k=1}^{+\infty} a_k) \Rightarrow I_n \text{ upper-bounded.} \quad (3)$$

From Eq.(2) and Eq.(3):

$$\begin{cases} I_n \text{ increasing} \\ I_n \text{ upper-bounded} \end{cases} \Rightarrow I_n \text{ convergent} \Rightarrow \exists l \in \mathbb{R} : \lim I_n = l.$$

We note that; from Eq.(1):

$$\forall n \in \mathbb{N}^* : \forall x \in [n, n+1] : I_n \leq \int_1^x f(t)\,dt \leq I_{n+1} \Rightarrow$$

$$\Rightarrow \lim I_n \leq \int_1^{+\infty} f(t)\,dt \leq \lim I_{n+1} \Rightarrow l \leq \int_1^{+\infty} f(t)\,dt \leq l$$

$$\Rightarrow \int_1^{+\infty} f(t)\,dt = l \Rightarrow \int_1^{+\infty} f(t)\,dt \text{ converges.}$$

($\Leftarrow$): Assume that $\int_1^{+\infty} f(x)dx$ converges and define $\forall n \in \mathbb{N}^*: S_n = \sum_{k=1}^{n} a_k$.

Since $(\forall n \in \mathbb{N}^* : a_n > 0)$, it follows that:

$$\forall n \in \mathbb{N}^*: S_{n+1} = \sum_{k=1}^{n+1} a_k = a_{n+1} + \sum_{k=1}^{n} a_k > \sum_{k=1}^{n} a_k = S_n$$

$\Rightarrow (\forall n \in \mathbb{N}^*: S_{n+1} > S_n) \Rightarrow S_n$ strictly increasing. (4)

We also note that

$$\forall n \in \mathbb{N}^*: S_n = \sum_{k=1}^{n} a_k = a_1 + \sum_{k=2}^{n} a_k = a_1 + I_n(f|1,n)$$

$$\leqslant a_1 + I_n = a_1 + \int_1^n f(x)dx \leqslant a_1 + \int_1^{+\infty} f(x)dx$$

$\Rightarrow S_n$ upper bounded. (5)

From Eq. (4) and Eq. (5):

$$\begin{cases} S_n \text{ strictly increasing} \Rightarrow S_n \text{ converges} \\ S_n \text{ upper bounded} \end{cases} \Rightarrow \sum_{k=1}^{+\infty} a_k \text{ converges.} \quad \square$$

↳ Application : Using the integral test, we can show that:

$$\sum_{k=1}^{+\infty} \frac{1}{k^p} \text{ converges} \iff p > 1$$

$$\sum_{k=1}^{+\infty} \frac{1}{k^p} \text{ diverges} \iff p \leq 1$$

- This result can be used in conjunction with forthcoming convergence tests (i.e. comparison test, limit comparison test) It also motivates the definition of the famous Riemann-Zeta function, defined as:

$$\forall s \in [1, +\infty) : \zeta(s) = \sum_{k=1}^{+\infty} \frac{1}{k^s}$$

- A remarkable result, originally shown by Euler, is that $\zeta(2) = \pi^2/6$.

Proof

We define $\forall x \in [1, +\infty) : f(x) = \frac{1}{x^p}$

It is easy to see that for $p < 0$:

$$\lim_{x \to +\infty} \frac{1}{x^p} = \lim_{x \to +\infty} x^{-p} = +\infty \Rightarrow \lim_{n \in \mathbb{N}^*} \frac{1}{n^p} \neq 0 \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{+\infty} \frac{1}{k^p} \text{ diverges}$$

and for $p=0$:

$$\lim_{x\to+\infty} \frac{1}{x^p} = \lim_{x\to+\infty} \frac{1}{x^0} = 1 \Rightarrow \lim_{n\in\mathbb{N}^*} \frac{1}{n^p} = 1 \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{+\infty} \frac{1}{k^p} \text{ diverges}$$

We may now assume, with no loss of generality, that $p>0$.
It follows that

$$\forall x\in[1,+\infty): f'(x) = \frac{d}{dx}\left(\frac{1}{x^p}\right) = (x^{-p})' = -px^{-p-1} =$$

$$= \frac{-p}{x^{p+1}} < 0$$

$$\Rightarrow f \searrow [1,+\infty) \qquad (1)$$

and also that

$$f \text{ continuous on } [1,+\infty) \qquad (2)$$

From Eq.(1) and Eq.(2), via the integral test, it follows that:

$$\sum_{k=1}^{+\infty} \frac{1}{k^p} \text{ converges} \Leftrightarrow \int_1^{+\infty} \frac{dx}{x^p} \text{ converges}$$

$$\Leftrightarrow p>1. \qquad \square$$

EXAMPLE

a) Show that $\sum_{n=1}^{+\infty} \frac{1}{(n+1)\ln(n+1)}$ diverges.

Solution

We define: $\forall x \in [1,+\infty): f(x) = \frac{1}{(x+1)\ln(x+1)}$

and note that

f continuous on $[1,+\infty)$ (1)

and

$$\forall x \in [1,+\infty): f'(x) = \frac{-[(x+1)\ln(x+1)]'}{[(x+1)\ln(x+1)]^2} =$$

$$= \frac{-[(x+1)'\ln(x+1) + (x+1)[\ln(x+1)]']}{(x+1)^2[\ln(x+1)]^2}$$

$$= \frac{-\ln(x+1) - (x+1)(1/(x+1))}{(x+1)^2[\ln(x+1)]^2} =$$

$$= \frac{-\ln(x+1) - 1}{(x+1)^2[\ln(x+1)]^2}$$

Let $x \in [1,+\infty)$ be given. Then:

$x \in [1,+\infty) \Rightarrow x \geq 1 \Rightarrow x+1 \geq 2 > 1 \Rightarrow \ln(x+1) > \ln 1 \Rightarrow$

$\Rightarrow -\ln(x+1) < 0 \Rightarrow -\ln(x+1) - 1 < 0 \Rightarrow$

$\Rightarrow f'(x) < 0$

since $(x+1)^2[\ln(x+1)]^2 > 0$. It follows that:

$(\forall x \in [1,+\infty): f'(x) < 0) \Rightarrow f \searrow [1,+\infty)$ (2)

Finally, we note that:

$$\int_1^x f(t)\,dt = \int_1^x \frac{dt}{(t+1)\ln(t+1)} = \int_2^{x+1} \frac{dt}{t\ln t} = \int_2^{x+1} \frac{(\ln t)'}{\ln t}\,dt$$

$$= \Big[\ln|\ln t|\Big]_2^{x+1} = \ln|\ln(x+1)| - \ln|\ln(2)|\,,\ \forall x \in [1,+\infty).$$

and that:

$$\lim_{x\to+\infty}(x+1) = \lim_{x\to+\infty} x = +\infty \Rightarrow \lim_{x\to+\infty} \ln(x+1) = +\infty \longrightarrow$$

$$\Rightarrow \lim_{x\to+\infty} |\ln(x+1)| = +\infty \Rightarrow \lim_{x\to+\infty} \ln|\ln(x+1)| = +\infty$$

$$\Rightarrow \lim_{x\to+\infty} \left[\ln|\ln(x+1)| - \ln|\ln 2|\right] = +\infty \Rightarrow$$

$$\Rightarrow \int_1^{+\infty} \frac{dx}{(x+1)\ln(x+1)} \quad \text{diverges} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{(n+1)\ln(n+1)} \quad \text{diverges.}$$

② → Comparison test

Thm : Let a_n, b_n be two sequences and assume that

$$\forall n \in \mathbb{N}^*: (n > n_0 \Rightarrow 0 \leq a_n \leq b_n)$$

Then,

$$\sum_{n=1}^{+\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges}$$

$$\sum_{n=1}^{+\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} b_n \text{ diverges}$$

Proof

We define: $\forall n \in \mathbb{N}^*: \left(A_n = \sum_{k=1}^{n} a_n \wedge B_n = \sum_{k=1}^{n} b_k\right)$

Let us assume that B_n converges. Then:

B_n converges $\Rightarrow$ B_n bounded

$\Rightarrow$ B_n upper bounded

$\Rightarrow \exists B > 0 : \forall n \in \mathbb{N}^* : B_n \leq B$

Now we will show that A_n is bounded.

Let $n \in \mathbb{N}^*$ be given, with $n > n_0$. Then:

$$A_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n_0} a_k + \sum_{k=n_0+1}^{n} a_k \leq \sum_{k=1}^{n_0} a_k + \sum_{k=n_0+1}^{n} b_k \leq$$

$$\leq \sum_{k=1}^{n_0} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n_0} a_k + B_n \leq \sum_{k=1}^{n_0} a_k + B = A_{n_0} + B$$

For $n \leq n_0$: $A_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n_0} a_k \leq \sum_{k=1}^{n_0} a_k + B = A_{n_0} + B$

It follows that:

$(\forall n \in \mathbb{N}^* : A_n \leq A_{n+1} + B) \Rightarrow A_n$ upper bounded. (1)

Let $n \in \mathbb{N}^*$ be given. Then:

$$A_{n+1} = \sum_{k=1}^{n+1} a_k \geq \sum_{k=1}^{n} a_k = A_n$$

and it follows that

$(\forall n \in \mathbb{N}^* : A_{n+1} \geq A_n) \Rightarrow A_n$ increasing (2)

From Eq. (1) and Eq. (2):

$$\begin{cases} A_n \text{ upper bounded} \Rightarrow A_n \text{ converges.} \\ A_n \text{ increasing} \end{cases}$$

An immediate consequence is the contrapositive statement

A_n diverges $\Rightarrow B_n$ diverges. □

EXAMPLES

a) Show that $\sum_{k=1}^{+\infty} \frac{1}{k!}$ converges.

Solution

We define $\forall n \in \mathbb{N}^* : a_n = 1/n! \geq 0$ (1)

and note that

$$\forall n \in \mathbb{N}^* - [2] : a_n = \frac{1}{n!} = \frac{1}{n(n-1)(n-2)!} \leq \frac{1}{n(n-1)}$$

$$\leq \frac{1}{(n-1)^2} \quad (2)$$

From Eq.(1) and Eq.(2)

$\forall n \in \mathbb{N}^* - [2] : 0 \leq a_n \leq 1/(n-1)^2$ (3)

From Eq.(3), via the comparison test, it follows that

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} \text{ converges} \Rightarrow \sum_{k=2}^{+\infty} \frac{1}{(k-1)^2} \text{ converges} \Rightarrow$$

$$\Rightarrow \sum_{k=2}^{+\infty} \frac{1}{k!} \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{k!} \text{ converges}$$

↳ Note that Eq.(2) is valid only for $n \geq 2$, which is needed to write $n! = n(n-1)(n-2)!$. It is therefore important to indicate that in the quantifier $\forall n \in \mathbb{N}^* - [2]$. Note that $\mathbb{N}^* - [2] = \{1, 2, 3, \ldots\} - \{1, 2\} = \{3, 4, 5, \ldots\}$

b) Show that $\sum_{n=1}^{+\infty} \frac{1}{2+n5^n}$ converges.

Solution

We define $\forall n \in \mathbb{N}^*: a_n = \frac{1}{2+n5^n}$

and note that: $\forall n \in \mathbb{N}^*: a_n \geq 0$ (1)

and

$$\forall n \in \mathbb{N}^*: a_n = \frac{1}{2+n5^n} \leq \frac{1}{n5^n} \leq \frac{1}{5^n} = \left(\frac{1}{5}\right)^n \quad (2)$$

From Eq.(1) and Eq.(2):

$\forall n \in \mathbb{N}^*: 0 \leq a_n \leq (1/5)^n$

and from the comparison test, it follows that

$\sum_{n=1}^{+\infty} (1/5)^n$ converges $\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{2+n5^n}$ converges.

c) Show that $\sum_{n=1}^{+\infty} \frac{n^2}{4n^3-3}$ diverges.

Solution

We define $\forall n \in \mathbb{N}^*: a_n = \frac{n^2}{4n^3-3}$.

Let $n \in \mathbb{N}^*$ be given. Since

$$a_n = \frac{n^2}{4n^3-3} \geq \frac{n^2}{4n^3} = \frac{1}{4n} \geq 0$$

It follows that

$$\forall n \in \mathbb{N}^*: \left(0 \leq \frac{1}{4n} \leq a_n\right) \qquad (1)$$

From Eq.(1), via the comparison theorem, it follows that

$$\sum_{n=1}^{+\infty} \frac{1}{n} \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} \frac{1}{4n} \text{ diverges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{n^2}{4n^3 - 3} \text{ diverges.} \quad \square$$

(3) → Limit Comparison Test

Thm: Let a_n, b_n be two sequences and assume that

$$\begin{cases} \forall n \in \mathbb{N}^*: (a_n \geq 0 \wedge b_n > 0) \\ \lim (a_n/b_n) = l \in \mathbb{R} \end{cases}$$

Then:

$$l > 0 \Rightarrow \left(\sum_{n=1}^{+\infty} a_n \text{ converges} \Leftrightarrow \sum_{n=1}^{+\infty} b_n \text{ converges} \right)$$

$$l = 0 \Rightarrow \left(\sum_{n=1}^{+\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges} \right)$$

and the corresponding contrapositive statements are:

$$l > 0 \Rightarrow \left(\sum_{n=1}^{+\infty} a_n \text{ diverges} \Leftrightarrow \sum_{n=1}^{+\infty} b_n \text{ diverges} \right)$$

$$l = 0 \Rightarrow \left(\sum_{n=1}^{+\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} b_n \text{ diverges} \right)$$

Proof

Case 1: Assume that $l > 0$. Then

$\lim (a_n/b_n) = l > 0 \Rightarrow$

$\Rightarrow \forall \varepsilon > 0: \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: |a_n/b_n - l| < \varepsilon$

Choose $\varepsilon = l/2$ and fix a $n_0 \in \mathbb{N}^*$ such that

$\forall n \in \mathbb{N}^* - [n_0]: |a_n/b_n - l| < l/2$

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then:

$|a_n/b_n - l| < l/2 \Rightarrow -l/2 < a_n/b_n - l < l/2 \Rightarrow$

$\Rightarrow l - l/2 < a_n/b_n < l + l/2 \Rightarrow$

$$\Rightarrow l/2 < a_n/b_n < 3l/2 \Rightarrow \begin{cases} l/2 < a_n/b_n \\ a_n/b_n < 3l/2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} b_n < (2/l)\,a_n \\ a_n < (3l/2)\,b_n \end{cases}$$

It follows that

$$\forall n \in \mathbb{N}^* - [n_0]: \begin{cases} b_n < (2/l)\,a_n \\ a_n < (3l/2)\,b_n \end{cases}$$

and therefore, from the comparison test:

$$\sum_{n=1}^{+\infty} a_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} (2/l)\,a_n \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} b_n \text{ converges}$$

and

$$\sum_{n=1}^{+\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} (3l/2)\,b_n \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.}$$

We have thus shown that

$$\sum_{n=1}^{+\infty} a_n \text{ converges} \Leftrightarrow \sum_{n=1}^{+\infty} b_n \text{ converges.}$$

Case 2: Assume that $l = 0$. Then

$$\lim(a_n/b_n) = 0 \Rightarrow$$

$$\Rightarrow \forall \varepsilon > 0: \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: |a_n/b_n| < \varepsilon$$

Choose any $\varepsilon > 0$ and fix $n_0 \in \mathbb{N}^*$ such that

$$\forall n \in \mathbb{N}^* - [n_0]: |a_n/b_n| < \varepsilon.$$

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then, since $a_n \geq 0$ and $b_n > 0$,

it follows that

$|a_n/b_n| < \varepsilon \Rightarrow a_n/b_n < \varepsilon \Rightarrow a_n < \varepsilon b_n$

and therefore:

$\forall n \in \mathbb{N}^* - [n_0] : 0 < a_n < \varepsilon b_n$

From the comparison test, it follows that

$\sum_{n=1}^{+\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{+\infty} \varepsilon b_n$ converges

$\Rightarrow \sum_{n=1}^{+\infty} a_n$ converges. ∎

EXAMPLE

a) Show that $\sum_{n=1}^{+\infty} \frac{3n-2}{n^3+2n^2+11}$ converges.

Solution

We define:

$$\forall n \in \mathbb{N}^*: \left(a_n = \frac{3n-2}{n^3+2n^2+11} \;\wedge\; b_n = \frac{1}{n^2} \right)$$

It follows that

$$\forall n \in \mathbb{N}^*: \frac{a_n}{b_n} = \frac{3n-2}{n^3+2n^2+11} \left(\frac{1}{n^2}\right)^{-1} = \frac{n^2(3n-2)}{n^3+2n^2+11} \quad (1)$$

and since

$$\lim_{x \to +\infty} \frac{x^2(3x-2)}{x^3+2x^2+11} = \lim_{x \to +\infty} \frac{3x^3}{x^3} = 3 \Rightarrow$$

$$\Rightarrow \lim \frac{a_n}{b_n} = \lim \frac{n^2(3n-2)}{n^3+2n^2-11} = 3 > 0 \quad (2)$$

it follows from the limit comparison test that

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{3n-2}{n^3+2n^2+11} \text{ converges.}$$

↳ Note that for large n we can reasonably expect a_n to scale as $3n/n^3 = 3/n^2$, which is why we are comparing it with $b_n = 1/n^2$.

b) Show that $\sum_{n=1}^{+\infty} \frac{n}{2^n(n+1)}$ converges.

Solution

We define:

$$\forall n \in \mathbb{N}^*: \left(a_n = \frac{n}{2^n(n+1)} \wedge b_n = \frac{1}{2^n} \right)$$

It follows that

$$\forall n \in \mathbb{N}^*: \frac{a_n}{b_n} = \frac{\frac{n}{2^n(n+1)}}{\frac{1}{2^n}} = \frac{2^n n}{2^n(n+1)} = \frac{n}{n+1}$$

and since:

$$\lim_{x \to +\infty} \frac{x}{x+1} = \lim_{x \to +\infty} \frac{x}{x} = 1 \Rightarrow \lim \frac{a_n}{b_n} = 1$$

and therefore, from the limit comparison theorem, it follows that

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{n}{2^n(n+1)} \text{ converges.}$$

c) Show that $\sum_{n=1}^{+\infty} \frac{\ln n}{n^2}$ converges

Solution

We define:

$$\forall n \in \mathbb{N}^*: \left(a_n = \frac{\ln n}{n^2} \wedge b_n = \frac{1}{n^s} \right)$$

and note that

$$\forall n \in \mathbb{N}^*: \quad \frac{a_n}{b_n} = \frac{\frac{\ln n}{n^2}}{\frac{1}{n^s}} = \frac{n^s \ln n}{n^2} = \frac{\ln n}{n^{2-s}}$$

Let us assume that $2-s>0$. Then:

$$\lim_{x\to+\infty} \frac{\ln x}{x^{2-s}} \overset{\infty/\infty}{=} \lim_{x\to+\infty} \frac{(\ln x)'}{(x^{2-s})'} = \lim_{x\to+\infty} \frac{x^{-1}}{(2-s)x^{1-s}} =$$

$$= \lim_{x\to+\infty} \frac{1}{(2-s)x^{2-s}} = \frac{1}{2-s} \cdot 0 = 0$$

$$\Rightarrow \lim \frac{a_n}{b_n} = 0 \quad , \forall s \in (-\infty, 2) \qquad (1)$$

For $s=3/2$, via the limit comparison test,

$$\sum_{n=1}^{+\infty} \frac{1}{n^{3/2}} \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{\ln n}{n^2} \text{ converges.} \quad \square$$

↳ Since it is not obvious what to compare with, we try the general case $b_n = 1/n^s$. Note that the De L'Hospital step requires $2-s>0$, otherwise the limit $x\to+\infty$ will not be a ∞/∞ limit. So the proof of Eq.(1) is valid only when $s<2$. Note that any choice $1<s<2$ will work as well as $s=3/2$. For $s\leq 1$ there is a problem. We cannot say

$$\sum_{n=1}^{+\infty} \frac{1}{n^s} \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} \frac{\ln n}{n^2} \text{ diverges}$$

because the implication is going in the wrong direction. Since $\lim (a_n/b_n) = 0$, the implication does not work in both directions. From the limit comparison test, we can only conclude:

$$\sum_{n=1}^{+\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges}$$

$$\sum_{n=1}^{+\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} b_n \text{ diverges.}$$

It is very important to apply the theorem correctly to avoid any errors in logic.

(4) → Ratio test

Let a_n be a sequence. Then:

$$\begin{cases} \forall n \in \mathbb{N}^*: a_n > 0 \\ \lim (a_{n+1}/a_n) = \rho < 1 \end{cases} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges}$$

$$\begin{cases} \forall n \in \mathbb{N}^*: a_n > 0 \\ \lim (a_{n+1}/a_n) = \rho > 1 \end{cases} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges}$$

Proof

▸ Part 1: Assume that $\rho < 1$. Since:

$$(\forall n \in \mathbb{N}^*: a_n > 0) \Rightarrow (\forall n \in \mathbb{N}^*: a_{n+1}/a_n > 0)$$
$$\Rightarrow \rho = \lim (a_{n+1}/a_n) \geq 0$$

thus, we know that $0 \leq \rho < 1$. Since

$\lim (a_{n+1}/a_n) = \rho \Rightarrow \forall \varepsilon > 0: \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: |a_{n+1}/a_n - \rho| < \varepsilon$.

Choose an $r \in \mathbb{R}$ such that $r \in (\rho, 1)$ and choose $\varepsilon = r - \rho > 0$.

We also fix $n_0 \in \mathbb{N}^*$ such that

$$\forall n \in \mathbb{N}^* - [n_0]: |a_{n+1}/a_n - \rho| < r - \rho$$

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then:

$$|a_{n+1}/a_n - \rho| < r - \rho \Rightarrow -(r-\rho) < (a_{n+1}/a_n) - \rho < r - \rho \Rightarrow$$
$$\Rightarrow (a_{n+1}/a_n) - \rho < r - \rho \Rightarrow a_{n+1}/a_n < r \Rightarrow a_{n+1} < r\, a_n$$

The last step uses the assumption $a_n > 0$. It follows that

$$\forall n \in \mathbb{N}^* - [n_0]: a_{n+1} < r a_n$$

▸ We claim that $\forall n \in \mathbb{N}^* - [n_0]: a_n < r^{n-n_0} a_{n_0}$ (1)

For $n = n_0 + 1$: $a_n = a_{n_0+1} < r a_{n_0} = r^{(n_0+1)-n_0} a_{n_0} = r^{n-n_0} a_{n_0}$

$$\Rightarrow a_n < r^{n-n_0} a_{n_0}$$

For $n = n_0 + k$ with $k \in \mathbb{N}^*$ we assume that

$$a_n < r^{n-n_0} a_{n_0} \Leftrightarrow a_{n_0+k} < r^{(n_0+k)-n_0} a_{n_0} \Leftrightarrow$$

$$\Leftrightarrow a_{n_0+k} < r^k a_{n_0}$$

For $n = n_0 + k + 1$ we show that

$$a_n = a_{n_0+k+1} < r a_{n_0+k} < r(r^k a_{n_0}) = r^{k+1} a_{n_0}$$

$$= r^{(n_0+k+1)-n_0} a_{n_0} = r^{n-n_0} a_{n_0} \Rightarrow a_n < r^{n-n_0} a_{n_0}$$

The claim is thus shown by induction.

From Eq. (1) and our assumptions we get

$$\forall n \in \mathbb{N}^* - [n_0] : 0 \leq a_n < r^{n-n_0} a_{n_0}$$

and, via the comparison test, it follows that

$$\sum_{n=n_0+1}^{+\infty} r^{n-n_0} \text{ converges} \Rightarrow \sum_{n=n_0+1}^{+\infty} r^{n-n_0} a_{n_0} \text{ converges}$$

$$\Rightarrow \sum_{n=n_0+1}^{+\infty} a_n \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.}$$

▸ Part 2 : Assume that $\rho > 1$. Then:

$$\lim (a_{n+1}/a_n) = \rho > 1 \Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : a_{n+1}/a_n > 1$$

Fix $n_0 \in \mathbb{N}^*$ such that $\forall n \in \mathbb{N}^* - [n_0] : a_{n+1}/a_n > 1$

Then $\forall n \in \mathbb{N}^* - [n_0] : a_{n+1} > a_n$

We claim that $\forall n \in \mathbb{N}^* - [n_0] : a_n \geq a_{n_0+1}$

For $n = n_0 + 1$: $a_n = a_{n_0+1} \Rightarrow a_n \geq a_{n_0+1}$

For $n = k > n_0 + 1$, we assume that $a_k \geq a_{n_0+1}$

For $n = k + 1$: $a_n = a_{k+1} > a_k \geq a_{n_0+1} \Rightarrow a_n \geq a_{n_0+1}$.

It follows, by induction, that
$(\forall n \in \mathbb{N}^* - [n_0]^0 : a_n \geq a_{n_0+1}) \Rightarrow$
$\Rightarrow \lim a_n \geq a_{n_0+1} \vee \lim a_n$ does not exist
$\Rightarrow \lim a_n \neq 0 \Rightarrow \sum_{n=1}^{+\infty} a_n$ diverges. □

↔ The ratio test is particularly useful in series involving the factorial or the double factorial, defined as follows:

1) Factorial : It is defined recursively by:

$$\begin{cases} 0! = 1 \\ n! = n(n-1)! \ , \ \forall n \in \mathbb{N}^* \end{cases}$$

or equivalently by

$$n! = \prod_{a=1}^{n} a = n(n-1) \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

2) Double factorial: It is defined recursively by

$$\begin{cases} 0!! = 1 \ \wedge \ 1!! = 1 \\ (2n)!! = 2n(2n-2)!! \ , \ \forall n \in \mathbb{N}^* \\ (2n+1)!! = (2n+1)(2n-1)!! \ , \ \forall n \in \mathbb{N}^* \end{cases}$$

or equivalently

$$(2n)!! = \prod_{a=1}^{n} 2a = 2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n) \ , \ \forall n \in \mathbb{N}^*$$

$$(2n+1)!! = \prod_{a=0}^{n} (2a+1) = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1) \ , \ \forall n \in \mathbb{N}$$

EXAMPLES

a) Show that $\sum_{n=1}^{+\infty} \frac{(n!)^2}{(2n)!}$ converges

Solution

We define

$$a_n = \frac{(n!)^2}{(2n)!} > 0, \quad \forall n \in \mathbb{N}^* \qquad (1)$$

and note that

$$\frac{a_{n+1}}{a_n} = \frac{\left[\frac{((n+1)!)^2}{(2n+2)!}\right]}{\left[\frac{(n!)^2}{(2n)!}\right]} = \frac{(2n)!\,((n+1)!)^2}{(n!)^2\,(2n+2)!} =$$

$$= \frac{(2n)!\,[(n+1)n!]^2}{(n!)^2\,(2n+2)(2n+1)(2n)!} = \frac{(n+1)^2}{(2n+2)(2n+1)}, \quad \forall n \in \mathbb{N}^*$$

Since

$$\lim_{x \to +\infty} \frac{(x+1)^2}{(2x+2)(2x+1)} = \lim_{x \to +\infty} \frac{x^2}{(2x)^2} = \lim_{x \to +\infty} \frac{x^2}{4x^2}$$

$$= \frac{1}{4} \Rightarrow$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} \frac{a_{n+1}}{a_n} = \lim_{n \in \mathbb{N}^*} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1 \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\sum_{n=1}^{+\infty} \frac{(n!)^2}{(2n)!} \text{ converges}$$

b) Show that $\sum_{n=1}^{+\infty} \frac{5^n}{(2n+1)!!}$ converges.

Solution

We define

$$a_n = \frac{5^n}{(2n+1)!!} > 0 \,, \quad \forall n \in \mathbb{N}^* \qquad (1)$$

and note that

$$\frac{a_{n+1}}{a_n} = \frac{\left[\frac{5^{n+1}}{(2n+3)!!}\right]}{\left[\frac{5^n}{(2n+1)!!}\right]} = \frac{5^{n+1}(2n+1)!!}{5^n (2n+3)!!} =$$

$$= \frac{5(2n+1)!!}{(2n+3)(2n+1)!!} = \frac{5}{2n+3} \,, \quad \forall n \in \mathbb{N}^*$$

Since

$$\lim_{x \to +\infty} \frac{5}{2x+3} = \lim_{x \to +\infty} \frac{5}{2x} = 0 \Rightarrow$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} \frac{a_{n+1}}{a_n} = \lim_{n \in \mathbb{N}^*} \frac{5}{2n+3} = 0 < 1 \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\sum_{n=1}^{+\infty} \frac{5^n}{(2n+1)!!} \quad \text{converges}$$

c) Show that $\sum_{n=1}^{+\infty} \frac{(3n+1)!}{2^n (n!)^2 (2n)!!}$ diverges.

Solution

We define $\forall n \in \mathbb{N}^* : a_n = \frac{(3n+1)!}{2^n (n!)^2 (2n)!!} > 0$ (1)

and note that

$$\frac{a_{n+1}}{a_n} = \frac{\left[\frac{(3n+4)!}{2^{n+1}((n+1)!)^2 (2n+2)!!}\right]}{\left[\frac{(3n+1)!}{2^n (n!)^2 (2n)!!}\right]} =$$

$$= \frac{2^n (n!)^2 (2n)!! (3n+4)!}{2^{n+1}((n+1)!)^2 (2n+2)!! (3n+1)!}$$

$$= \frac{2^n (n!)^2 (2n)!! (3n+4)(3n+3)(3n+2)(3n+1)!}{2 \cdot 2^n (n+1)^2 (n!)^2 (2n+2)(2n)!! (3n+1)!}$$

$$= \frac{(3n+4)(3n+3)(3n+2)}{2(n+1)^2 (2n+2)}, \quad \forall n \in \mathbb{N}^*$$

It follows that

$$\lim_{x \to +\infty} \frac{(3x+4)(3x+3)(3x+2)}{2(x+1)^2(2x+2)} = \lim_{x \to +\infty} \frac{(3x)^3}{2x^2 (2x)} =$$

$$= \lim_{x \to +\infty} \frac{27x^3}{4x^3} = \frac{27}{4} \Rightarrow \lim_{n \in \mathbb{N}^*} \frac{a_{n+1}}{a_n} = \frac{27}{4} > 1 \quad (2)$$

From Eq. (1) and Eq. (2) via the ratio test:

$$\sum_{n=1}^{+\infty} \frac{(3n+1)!}{2^n (n!)^2 (2n)!!} \text{ diverges}$$

(5) → Root test

Thm: Let a_n be a sequence. Then

$$\begin{cases} \forall n \in \mathbb{N}^* : a_n > 0 \\ \lim (a_n)^{1/n} = \rho < 1 \end{cases} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges}$$

$$\begin{cases} \forall n \in \mathbb{N}^* : a_n > 0 \\ \lim (a_n)^{1/n} = \rho > 1 \end{cases} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges}$$

Proof

Case 1 : Assume that $\rho < 1$. Since

$\lim (a_n)^{1/n} = \rho \Rightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n^{1/n} - \rho| < \varepsilon$

Choose an $r \in [\rho, 1)$ and choose $\varepsilon = r - \rho > 0$. Fix an $n_0 \in \mathbb{N}^*$ such that

$\forall n \in \mathbb{N}^* - [n_0] : |a_n^{1/n} - \rho| < r - \rho$.

Let $n \in \mathbb{N}^* - [n_0]$ be given. We note that $a_n > 0$ and also

$|a_n^{1/n} - \rho| < r - \rho \Rightarrow -(r-\rho) < a_n^{1/n} - \rho < r - \rho \Rightarrow$

$\Rightarrow a_n^{1/n} - \rho < r - \rho \Rightarrow a_n^{1/n} < r$

$\Rightarrow a_n < r^n$.

It follows that $\forall n \in \mathbb{N}^* - [n_0] : 0 < a_n < r^n$.

and from the comparison test, we have:

$0 < r < 1 \Rightarrow \sum_{n=n_0+1}^{+\infty} r^n$ converges $\Rightarrow \sum_{n=n_0+1}^{+\infty} a_n$ converges

$\Rightarrow \sum_{n=1}^{+\infty} a_n$ converges.

Case 2 : Assume that $\rho > 1$. Choose an $r \in [1, \rho]$. Then

$\lim (a_n)^{1/n} = \rho > r \Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : (a_n)^{1/n} > r$

Fix $n_0 \in \mathbb{N}^*$ such that $\forall n \in \mathbb{N}^* - [n_0] : (a_n)^{1/n} > r$

Then $\forall n \in \mathbb{N}^* - [n_0] : a_n > r^n > 0$

and from the comparison test, it follows that

$r > 1 \Rightarrow \sum_{n=n_0+1}^{+\infty} r^n$ diverges $\Rightarrow \sum_{n=n_0+1}^{+\infty} a_n$ diverges

$\rightarrow \sum_{n=1}^{+\infty} a_n$ diverges. $\square$

EXAMPLES

a) Show that $\sum_{n=1}^{+\infty} \left(\frac{3n^2+2n+1}{n^2+5n+2} \right)^n$ diverges.

Solution

Define $\forall n \in \mathbb{N}^*: a_n = \left(\frac{3n^2+2n+1}{n^2+5n+2} \right)^n$

and we note that $\forall n \in \mathbb{N}^*: a_n > 0$ (1)

and

$$\forall n \in \mathbb{N}^*: a_n^{1/n} = \left[\left(\frac{3n^2+2n+1}{n^2+5n+2} \right)^n \right]^{1/n} = \frac{3n^2+2n+1}{n^2+5n+2}$$

Since:

$$\lim_{x \to +\infty} \frac{3x^2+2x+1}{x^2+5x+2} = \lim_{x \to +\infty} \frac{3x^2}{x^2} = 3 \Rightarrow$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} a_n^{1/n} = \lim_{n \in \mathbb{N}^*} \frac{3n^2+2n+1}{n^2+5n+2} = 3 > 1 \quad (2)$$

From Eq.(1) and Eq.(2), via the root test, it follows that

$$\sum_{n=1}^{+\infty} \left(\frac{3n^2+2n+1}{n^2+5n+2} \right)^n \text{ diverges.}$$

b) Show that $\sum_{n=2}^{+\infty} \left(\frac{1}{\ln n} \right)^n$ converges

Solution

Define $\forall n \in \mathbb{N}-\{0,1\}: a_n = \left(\frac{1}{\ln n}\right)^n$

and we note that $\forall n \in \mathbb{N}-\{0,1\}: a_n > 0$ (1)

and

$$\forall n \in \mathbb{N}-\{0,1\}: a_n^{1/n} = \left[\left(\frac{1}{\ln n}\right)^n\right]^{1/n} = \frac{1}{\ln n}$$

Since

$$\lim_{x\to+\infty} \ln x = +\infty \Rightarrow \lim_{x\to+\infty} \frac{1}{\ln x} = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}-\{0,1\}} a_n^{1/n} = \lim_{n\in\mathbb{N}-\{0,1\}} \frac{1}{\ln n} = 0 \quad (2)$$

From Eq.(1) and Eq.(2): $\sum_{n=2}^{+\infty} \left(\frac{1}{\ln n}\right)^n$ converges.

c) Show that $\sum_{n=0}^{+\infty} n^3 (1/2)^n$ converges

Solution

Define $\forall n \in \mathbb{N}^*: a_n = n^3 (1/2)^n > 0$. It follows that

$$\lim_{n\in\mathbb{N}^*} \sqrt[n]{a_n} = \lim_{n\in\mathbb{N}^*} \sqrt[n]{n^3 (1/2)^n} = \lim_{n\in\mathbb{N}^*} \left[(1/2)\sqrt[n]{n^3}\right]$$

$$= (1/2) \lim_{n\in\mathbb{N}^*} \left(\sqrt[n]{n}\right)^3 = (1/2)\cdot 1 < 1 \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} n^3 (1/2)^n \text{ converges} \Rightarrow \sum_{n=0}^{+\infty} n^3 (1/2)^n \text{ converges.}$$

Other convergence tests

- Let a_n be a sequence. An alternating series, is a series of the form

$$s_n = \sum_{k=1}^{+\infty} (-1)^k a_k$$

with $\forall k \in \mathbb{N}^*: a_k \geq 0$

(1) → Leibnitz Rule (Alternating series test)

Thm: Let a_n be a sequence. Then

$$\left.\begin{array}{l} a_n \text{ decreasing} \\ \forall n \in \mathbb{N}^*: a_n \geq 0 \\ \lim a_n = 0 \end{array}\right\} \Rightarrow \sum_{n=1}^{+\infty} (-1)^n a_n \text{ converges}$$

Proof

We define $\forall n \in \mathbb{N}^*: s_n = \sum_{k=1}^{n} (-1)^k a_k$

▶ We claim that s_{2n} is decreasing and lower-bounded.

Let $n \in \mathbb{N}^*$ be given. We note that

$$s_{2n+2} = \sum_{k=1}^{2n+2} (-1)^k a_k = \sum_{k=1}^{2n} (-1)^k a_k + (-1)^{2n+1} a_{2n+1} + (-1)^{2n+2} a_{2n+2}$$

$$= s_{2n} + (a_{2n+2} - a_{2n+1}) \leq s_{2n}$$

Because a_n decreasing $\Rightarrow a_{2n+2} \leq a_{2n+1} \Rightarrow$

$$\Rightarrow a_{2n+2} - a_{2n+1} \leq 0$$

We also note that

$$s_{2n} = \sum_{k=1}^{2n} (-1)^k a_k =$$

$$= (-1)^1 a_1 + \sum_{k=2}^{2n-1} (-1)^k a_k + (-1)^{2n} a_{2n} =$$

$$= -a_1 + \sum_{k=1}^{n-1} \left((-1)^{2k} a_{2k} + (-1)^{2k+1} a_{2k+1} \right) + a_{2n}$$

$$= -a_1 + \sum_{k=1}^{n-1} (a_{2k} - a_{2k+1}) + a_{2n}$$

Since $(\forall n \in \mathbb{N}^* : a_k \geq 0) \Rightarrow a_{2n} \geq 0$

and since

a_n decreasing $\Rightarrow (\forall k \in \mathbb{N}^* : a_{2k} \geq a_{2k+1})$

$\Rightarrow (\forall k \in \mathbb{N}^* : a_{2k} - a_{2k+1} \geq 0)$

$\Rightarrow \sum_{k=1}^{n-1} (a_{2k} - a_{2k+1}) \geq 0$

and it follows that $s_{2n} \geq -a_1$

From the above it follows that

$\left\{ \begin{array}{l} \forall n \in \mathbb{N}^* : s_{2n+2} \leq s_{2n} \\ \forall n \in \mathbb{N}^* : s_{2n} \geq -a_1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} s_{2n} \text{ decreasing} \\ s_{2n} \text{ lower-bounded} \end{array} \right. \Rightarrow$

$\Rightarrow s_{2n}$ converges $\Rightarrow \exists l \in \mathbb{R} : \lim_{n \in \mathbb{N}^*} s_{2n} = l.$ (1)

We also note that

$$\lim s_{2n+1} = \lim (s_{2n} + (-1)^{2n+1} a_{2n+1}) = \lim (s_{2n} - a_{2n+1})$$

$$= \lim s_{2n} - \lim a_{2n+1} = l - 0 = l \quad (2)$$

From Eq. (1) and Eq. (2):

$\lim s_n = l \Rightarrow \sum_{n=1}^{+\infty} (-1)^n a_n$ converges. $\square$

↳ Note that in order to apply the alternating series test, it is important to show that the sequence a_n is decreasing. Oftentimes, this can be established via Calculus 1 methods.

EXAMPLES

a) Show that the sequence $\sum_{n=1}^{+\infty} (-1)^n \frac{n^2}{n^3+1}$ converges

Solution

We define:

$$\left(\forall n \in \mathbb{N}^* : a_n = \frac{n^2}{n^3+1}\right) \wedge \left(\forall x \in [1,+\infty) : f(x) = \frac{x^2}{x^3+1}\right)$$

and we note that:

$$\forall n \in \mathbb{N}^* : a_n \geq 0 \qquad (1)$$

and since

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2}{x^3+1} = \lim_{x\to+\infty} \frac{x^2}{x^3} = \lim_{x\to+\infty} \frac{1}{x} = 0$$

$$\Rightarrow \lim a_n = 0. \qquad (2)$$

We also note that:

$$\forall x \in [1,+\infty) : f'(x) = \frac{(x^2)'(x^3+1) - x^2(x^3+1)'}{(x^3+1)^2} =$$

$$= \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2} = \frac{x[2(x^3+1) - 3x^3]}{(x^3+1)^2}$$

$$= \frac{x(2x^3+2-3x^3)}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

and since:

$\forall x \in [2,+\infty) : (x>0 \wedge 2-x^3<0 \wedge (x^3+1)^2>0)$

$\Rightarrow (\forall x \in [2,+\infty) : f'(x)<0) \Rightarrow f$ strictly decreasing on $[2,+\infty)$

$\Rightarrow (\forall n \in \mathbb{N}^* - \{1\} : a_{n+1} < a_n) \Rightarrow a_n$

$\Rightarrow a_n$ decreasing on $\mathbb{N}^* - \{1\}$ (3)

From Eq.(1) and Eq.(2) and Eq.(3); via the alternating series test, we have:

$$\sum_{n=2}^{+\infty} (-1)^n a_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} (-1)^n \frac{n^2}{n^3+1} \text{ converges.}$$

b) Show that $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ converges

Solution

We define $\forall n \in \mathbb{N}^* : a_n = \frac{1}{n}$

and note that

$\forall n \in \mathbb{N}^* : a_n > 0$ (1)

Claim: We will show that a_n decreasing.

Let $n \in \mathbb{N}^*$ be given. Then:

$$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{n-(n+1)}{n(n+1)} = \frac{n-n-1}{n(n+1)}$$

$$= \frac{-1}{n(n+1)} < 0 \Rightarrow a_{n+1} < a_n$$

It follows that

$(\forall n \in \mathbb{N}^* : a_{n+1} < a_n) \Rightarrow a_n$ decreasing (2)

which proves the claim.

Finally we have

$$\lim_{x \to +\infty} \frac{1}{x} = 0 \Rightarrow \lim a_n = \lim \frac{1}{n} = 0 \quad (3)$$

From Eq.(1), Eq.(2), Eq.(3), via the alternating series test, it follows that

$$\sum_{n=1}^{+\infty} (-1)^n \frac{1}{n} \text{ converges.}$$

c) Show that $\sum_{n=2}^{+\infty} (-1)^n \frac{n^2+1}{n^2-1}$ diverges.

Solution

We define:

$$\forall n \in \mathbb{N}^* \setminus \{1\} : a_n = \frac{n^2+1}{n^2-1} (-1)^n$$

and therefore

$$\forall n \in \mathbb{N}^* : a_{2n} = (-1)^{2n} \frac{(2n)^2+1}{(2n)^2-1} = \frac{4n^2+1}{4n^2-1}$$

$$\forall n \in \mathbb{N}^* : a_{2n+1} = (-1)^{2n+1} \frac{(2n+1)^2+1}{(2n+1)^2-1} = \frac{-(4n^2+4n+1+1)}{4n^2+4n+1-1}$$

$$= \frac{-(4n^2+4n+2)}{4n^2+4n}$$

Since $\lim_{x \to +\infty} \frac{4x^2+1}{4x^2-1} = \lim_{x \to +\infty} \frac{4x^2}{4x^2} = 1 \Rightarrow \lim a_{2n} = 1$ (1)

and

$$\lim_{x\to+\infty} \frac{-(4x^2+4x+2)}{4x^2+4x} = \lim_{x\to+\infty} \frac{-4x^2}{4x^2} = -1 \Rightarrow \lim a_{2u+1} = -1 \quad (2)$$

From Eq.(1) and Eq.(2):

a_n diverges $\Rightarrow \lim a_n \neq 0 \Rightarrow \sum_{n=1}^{+\infty} (-1)^n \frac{n^2+1}{n^2-1}$ diverges. □

↳ One method for showing that an alternating series $\sum_{n=1}^{+\infty} (-1)^n a_n$ diverges is to show that $\lim (-1)^n a_n \neq 0$.

(2) → Absolute convergence test

Thm: Let a_n be a sequence. Then:

$$\sum_{n=1}^{+\infty} |a_n| \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.}$$

Proof

Let $n \in \mathbb{N}^*$ be given. Then:

$$-|a_n| \leq a_n \leq |a_n| \Rightarrow \begin{cases} -|a_n| \leq a_n \\ a_n \leq |a_n| \end{cases} \Rightarrow \begin{cases} a_n + |a_n| \geq 0 \\ a_n + |a_n| \leq 2|a_n| \end{cases}$$

$$\Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|$$

and it follows that

$$\forall n \in \mathbb{N}^* : 0 \leq a_n + |a_n| \leq 2|a_n|$$

From the comparison test, it follows that

$$\sum_{n=1}^{+\infty} |a_n| \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} 2|a_n| \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} (a_n + |a_n|) \text{ converges}$$

and since:

$$\begin{cases} \sum_{n=1}^{+\infty} (a_n + |a_n|) \text{ converges} \\ \sum_{n=1}^{+\infty} |a_n| \text{ converges} \end{cases} \Rightarrow \sum_{n=1}^{+\infty} [(a_n + |a_n|) - |a_n|] \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges} \quad \square$$

↳ Application to alternating series: If we show that $\sum_{n=1}^{+\infty} a_n$ converges, the absolute convergence test can be used to show $\sum_{n=1}^{+\infty} (-1)^n a_n$ converges.

EXAMPLES

a) Show that $\sum_{n=1}^{+\infty} (-1)^n \frac{n^2+3n+1}{2n^4+5n-1}$ converges.

Solution

We define $\forall n \in \mathbb{N}^*: \left(a_n = \frac{n^2+3n+1}{2n^4+5n-1} \wedge b_n = \frac{1}{n^2} \right)$

and note that

$$\forall n \in \mathbb{N}^*: \frac{a_n}{b_n} = \frac{\left(\frac{n^2+3n+1}{2n^4+5n-1}\right)}{\frac{1}{n^2}} = \frac{n^2(n^2+3n+1)}{2n^4+5n-1} =$$

$$= \frac{n^4+3n^3+n^2}{2n^4+5n-1}$$

Since $\lim_{x \to +\infty} \frac{x^4+3x^3+x^2}{2x^4+5x-1} = \lim_{x \to +\infty} \frac{x^4}{2x^4} = \frac{1}{2} \Rightarrow \lim_{n \in \mathbb{N}^*} \frac{a_n}{b_n} = \frac{1}{2}$

it follows from the limit comparison test that:

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{n^2+3n+1}{2n^4+5n-1} \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \left| (-1)^n \frac{n^2+3n+1}{2n^4+5n-1} \right| \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} (-1)^n \frac{n^2+3n+1}{2n^4+5n-1} \text{ converges.}$$

b) Show that $\sum_{n=1}^{+\infty} (-1)^n \frac{n^3}{3^n}$ converges.

Solution

Define $\forall n \in \mathbb{N}^*: a_n = \frac{n^3}{3^n}$

and note that

$$\forall n \in \mathbb{N}^*: \frac{a_{n+1}}{a_n} = \frac{\left(\frac{(n+1)^3}{3^{n+1}}\right)}{\left(\frac{n^3}{3^n}\right)} = \frac{3^n (n+1)^3}{3^{n+1} n^3} = \frac{3^n (n+1)^3}{3^n 3 n^3} =$$

$$= \frac{n^3+3n^2+3n+1}{3n^3}$$

Since

$$\lim_{x \to +\infty} \frac{x^3+3x^2+3x+1}{3x^3} = \lim_{x \to +\infty} \frac{x^3}{3x^3} = \frac{1}{3} \Rightarrow$$

$$\Rightarrow \lim \frac{a_{n+1}}{a_n} = \frac{1}{3} \Rightarrow \sum_{n=1}^{+\infty} \frac{n^3}{3^n} \text{ converges} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} \left| (-1)^n \frac{n^3}{3^n} \right| \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} (-1)^n \frac{n^3}{3^n} \text{ converges.} \quad \square$$

↳ The absolute convergence test can be combined with the comparison test to show that series of the form $\sum_{n=1}^{+\infty} a_n b_n$, with a_n bounded and $\sum_{n=1}^{+\infty} |b_n|$ convergent, also converge. Such arguments use the properties of absolute values:

$\forall a,b \in \mathbb{R}: |a+b| \leq |a|+|b|$ $\quad \forall a,b \in \mathbb{R}: |ab| = |a||b|$

$\forall a,b \in \mathbb{R}: |a-b| \leq |a|+|b|$ $\quad \forall a,b \in \mathbb{R}: \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

and the following inequalities:

$\forall x \in \mathbb{R}: |\sin x| \leq 1$

$\forall x \in \mathbb{R}: |\cos x| \leq 1$

$\forall x \in [-1,1]: |\operatorname{Arcsin}(x)| \leq \pi/2$

$\forall x \in \mathbb{R}: |\operatorname{Arctan} x| < \pi/2$

c) Show that $\sum_{n=1}^{+\infty} \frac{n[\sin(n!)+\cos(n!)]}{n^3+1}$ converges.

Solution

We define $\forall n \in \mathbb{N}^*: a_n = \frac{n[\sin(n!)+\cos(n!)]}{n^3+1}$

Let $n \in \mathbb{N}^*$ be given and since

$$|a_n| = \left|\frac{n[\sin(n!)+\cos(n!)]}{n^3+1}\right| = \frac{|n[\sin(n!)+\cos(n!)]|}{|n^3+1|}$$

$$= \frac{n}{n^3+1} \cdot |\sin(n!)+\cos(n!)|$$

$$\leq \frac{n(|\sin(n!)|+|\cos(n!)|)}{n^3+1} \leq \frac{n(1+1)}{n^3+1} =$$

$$= \frac{2n}{n^3+1} \leq \frac{2n}{n^3} = \frac{2}{n^2}$$

it follows that $\forall n \in \mathbb{N}^*: 0 \leq |a_n| \leq \frac{2}{n^2}$

and therefore, via the comparison test and the absolute convergence test:

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{2}{n^2} \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} |a_n| \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.} \qquad \square$$

d) Show that $\sum_{n=1}^{+\infty} \frac{n^3 \sin(\pi n/4) - n^2 \cos(\pi n/3)}{5^n}$ converges

Solution

We define $\forall n \in \mathbb{N}^*$: $a_n = \frac{n^3 \sin(\pi n/4) - n^2 \cos(\pi n/3)}{5^n}$

Let $n \in \mathbb{N}^*$ be given. Then

$$|a_n| = \left| \frac{n^3 \sin(\pi n/4) - n^2 \cos(\pi n/3)}{5^n} \right| =$$

$$= \frac{|n^3 \sin(\pi n/4) - n^2 \cos(\pi n/3)|}{5^n}$$

$$\leq \frac{|n^3 \sin(\pi n/4)| + |n^2 \cos(\pi n/3)|}{5^n} =$$

$$= \frac{n^3 |\sin(\pi n/4)| + n^2 |\cos(\pi n/3)|}{5^n}$$

$$\leq \frac{n^3 + n^2}{5^n} \leq \frac{n^3 + n^3}{5^n} = \frac{2n^3}{5^n}$$

and it follows that

$$\forall n \in \mathbb{N}^*: 0 \leq |a_n| \leq \frac{2n^3}{5^n} \qquad (1)$$

Define $\forall n \in \mathbb{N}^*: b_n = \frac{2n^3}{5^n}$

and note that

$$\forall n \in \mathbb{N}^*: \frac{b_{n+1}}{b_n} = \frac{\left(\frac{2(n+1)^3}{5^{n+1}}\right)}{\left(\frac{2n^3}{5^n}\right)} = \frac{2 \cdot 5^n (n+1)^3}{5^{n+1}(2n^3)} =$$

$$= \frac{(n+1)^3}{5n^3}$$

Since:

$$\lim_{x \to +\infty} \frac{(x+1)^3}{5x^3} = \lim_{x \to +\infty} \frac{x^3+3x^2+3x+1}{5x^3} =$$

$$= \lim_{x \to +\infty} \frac{x^3}{5x^3} = \frac{1}{5} \Rightarrow$$

$$\Rightarrow \lim \frac{b_{n+1}}{b_n} = \frac{1}{5} < 1 \Rightarrow \sum_{n=1}^{+\infty} \frac{2n^3}{5^n} \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} |a_n| \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.}$$

Conditional convergence

Note that the contrapositive of the absolute convergence test reads:

$$\sum_{n=1}^{+\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} |a_n| \text{ diverges}$$

However the converse of this statement, which is given by

$$\sum_{n=1}^{+\infty} |a_n| \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges}$$

is NOT always true and cannot be used in the solution of problems. Nevertheless, the absolute ratio test and absolute root test allow the formulation of similar correct arguments. We also note that the above observation motivates the following definition of conditional convergence:

Def: Let a_n be a sequence. Then:

$$\sum_{n=1}^{+\infty} a_n \text{ conditionally converges} \Leftrightarrow \begin{cases} \sum_{n=1}^{+\infty} a_n \text{ converges} \\ \sum_{n=1}^{+\infty} |a_n| \text{ diverges.} \end{cases}$$

An example of a conditionally convergent series is:

$$S_n = \sum_{k=1}^{n} (-1)^k \frac{1}{k}$$

The proof of the absolute ratio test and the absolute root test is based on the following lemma:

Lemma : Let a_n be a sequence. Then:

$$(\forall n \in \mathbb{N}^* - [n_0] : |a_n| \geq b > 0) \Rightarrow \lim a_n \neq 0$$

Proof

It is sufficient to show that

$$\exists \varepsilon > 0 : \forall n_1 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_1] : |a_n| \geq \varepsilon.$$

We choose $\varepsilon = b > 0$. Let $n_1 \in \mathbb{N}^*$ be given and choose $n = \max\{n_1 + 1, n_0 + 1\}$. Then:

$$n = \max\{n_1 + 1, n_0 + 1\} \Rightarrow n \geq n_0 + 1 \Rightarrow n \in \mathbb{N}^* - [n_0]$$
$$\Rightarrow |a_n| \geq b = \varepsilon \Rightarrow |a_n| \geq \varepsilon$$

It follows that

$$(\exists \varepsilon > 0 : \forall n_1 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_1] : |a_n| \geq \varepsilon) \Rightarrow \lim a_n \neq 0.$$

(3) → Absolute ratio test

Thm : Let a_n be a sequence. Then:

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r > 1 \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges}$$

Proof

Since:

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r > 1 \Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \left| \frac{a_{n+1}}{a_n} \right| > 1.$$

Fix $n_0 \in \mathbb{N}^*$ such that $\forall n \in \mathbb{N}^* - [n_0] : |a_{n+1}/a_n| > 1$.

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then:

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \frac{|a_{n+1}|}{|a_n|} > 1 \Rightarrow |a_{n+1}| > |a_n| > 0$$

It follows that:

$$(\forall n \in \mathbb{N}^* - [n_0] : |a_{n+1}| > |a_n|) \Rightarrow$$
$$\Rightarrow (\forall n \in \mathbb{N}^* - [n_0] : |a_n| \geq |a_{n_0+1}| > 0)$$
$$\Rightarrow \lim a_n \neq 0 \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges.} \quad \square$$

(4) → Absolute root test

Thm: Let a_n be a sequence. Then:

$$\lim |a_n|^{1/n} > r > 1 \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges}$$

Proof

Since $\lim |a_n|^{1/n} > r > 1 \Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n|^{1/n} > r > 1$

Fix $n_0 \in \mathbb{N}^*$ such that $\forall n \in \mathbb{N}^* - [n_0] : |a_n|^{1/n} > r > 1$

and let $n \in \mathbb{N}^* - [n_0]$ be given. Then:

$|a_n|^{1/n} > r \Rightarrow |a_n| = (|a_n|^{1/n})^n > r^n > r > 0$

It follows that

$$(\forall n \in \mathbb{N}^* - [n_0] : |a_n| > r > 0) \Rightarrow \lim a_n \neq 0 \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges.} \quad \square$$

EXAMPLES

a) Show that $\sum_{n=1}^{+\infty} (-1)^n \left[\frac{3n^2+n}{n^2+2}\right]^n$ diverges

Solution

We define $\forall n \in \mathbb{N}^*: a_n = (-1)^n \left[\frac{3n^2+n}{n^2+2}\right]^n$.

Then:

$$\forall n \in \mathbb{N}^*: |a_n|^{1/n} = \left|(-1)^n \left(\frac{3n^2+n}{n^2+2}\right)^n\right|^{1/n} =$$

$$= \left[\left(\frac{3n^2+n}{n^2+2}\right)^n\right]^{1/n} = \frac{3n^2+n}{n^2+2} \quad (1)$$

and we note that:

$$\lim_{x\to+\infty} \frac{3x^2+x}{x^2+2} = \lim_{x\to+\infty} \frac{3x^2}{x^2} = 3 \Rightarrow$$

$$\Rightarrow \lim |a_n|^{1/n} = 3 \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges.}$$

B) Show that $\sum_{n=1}^{+\infty} (-1)^n \frac{(n+1)^n}{n!}$ diverges.

Solution

We define $\forall n \in \mathbb{N}^*: a_n = (-1)^n \frac{(n+1)^n}{n!}$

and we note that

$$\forall n \in \mathbb{N}^*: \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+2)^{n+1}}{(n+1)!}}{\frac{(-1)^n (n+1)^n}{n!}}\right| = \frac{\frac{(n+2)^{n+1}}{(n+1)!}}{\frac{(n+1)^n}{n!}} =$$

$$= \frac{n!\,(n+2)^{n+1}}{(n+1)!\,(n+1)^n} = \frac{n!\,(n+2)^{n+1}}{(n+1)n!\,(n+1)^n} = \frac{(n+2)^{n+1}}{(n+1)^{n+1}} =$$

$$= \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

It follows that

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\left(1+\frac{1}{n+1}\right)^{n+1} = \lim\left(1+\frac{1}{n}\right)^{n} = e > 1$$

$$\Rightarrow \sum_{n=1}^{+\infty} (-1)^n \frac{(n+1)^n}{n!} \quad \text{diverges.}$$

CAL 2.6: Power Series

SERIES APPROXIMATION OF FUNCTIONS

Power series

- A power series is a series of the form
$$f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n \quad , \forall\, x \in A$$
with $\forall n \in \mathbb{N}^*$: $a_n \in \mathbb{R}$ and $x_0 \in \mathbb{R}$. The default domain A is the set of all $x \in \mathbb{R}$ for which the power series converges.

↳ Methodology: To establish convergence of a power series we use the ratio test or root test in conjunction with the absolute convergence test. To establish divergence, we use the absolute ratio test or the absolute root test.

EXAMPLES

a) Find the domain of $f(x) = \sum_{n=1}^{+\infty} \frac{(x-2)^n}{n!}$

Solution: Let $x \in \mathbb{R}$ be given.
Define $\forall n \in \mathbb{N}^*$: $a_n = \frac{(x-2)^n}{n!}$. Then:

$$\forall n \in \mathbb{N}^*: \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(x-2)^{n+1}}{(n+1)!}}{\frac{(x-2)^n}{n!}} \right| = \left| \frac{n!\,(x-2)^{n+1}}{(n+1)!\,(x-2)^n} \right| =$$

$$= \left| \frac{n!\,(x-2)(x-2)^n}{(n+1)\,n!\,(x-2)^n} \right| = \left| \frac{x-2}{n+1} \right| = \frac{|x-2|}{n+1}$$

Since:

$$\lim_{t\to+\infty} \frac{1}{t+1} = \lim_{t\to+\infty} \frac{1}{t} = 0 \Rightarrow \lim_{n\in\mathbb{N}^*} \frac{1}{n+1} = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \frac{|x-2|}{n+1} = 0 \Rightarrow \lim \left| \frac{a_{n+1}}{a_n} \right| = 0 \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} \left| \frac{(x-2)^n}{n!} \right| \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{(x-2)^n}{n!} \text{ converges.}$$

It follows that $\forall x\in\mathbb{R}: \left(\sum_{n=1}^{+\infty} \frac{(x-2)^n}{n!} \text{ converges} \right)$ and therefore $A=\mathbb{R}$.

B) Find the domain of $f(x) = \sum_{n=1}^{+\infty} n^2 3^n (x-2)^n$

Solution

Let $x\in\mathbb{R}$ be given, and define $\forall n\in\mathbb{N}^*: a_n = n^2 3^n (x-2)^n$.

Then,

$$\forall n\in\mathbb{N}^*: |a_n|^{1/n} = |n^2 3^n (x-2)^n|^{1/n} = 3\sqrt[n]{n^2}\,|x-2|$$

$$= 3\left(\sqrt[n]{n}\right)^2 |x-2|$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} |a_n|^{1/n} = \lim_{n\in\mathbb{N}^*} \left[3\left(\sqrt[n]{n}\right)^2 |x-2|\right] = 3|x-2| \lim_{n\in\mathbb{N}^*} \left(\sqrt[n]{n}\right)^2$$

$$= 3|x-2| \left(\lim \sqrt[n]{n}\right)^2 = 3|x-2| \cdot 1^2 =$$

$$= 3|x-2|.$$

Since,

$3|x-2|<1 \Leftrightarrow |x-2|<1/3 \Leftrightarrow -1/3<x-2<1/3 \Leftrightarrow$

$\Leftrightarrow -1/3+2<x<1/3+2 \Leftrightarrow 5/3<x<7/3$

we distinguish between the following cases:

Case 1 : Assume that $x \in (5/3, 7/3)$. Then:

$5/3<x<7/3 \Rightarrow 3|x-2|<1 \rightarrow \lim_{n\in\mathbb{N}^*} |a_n|^{1/n} < 1 \Rightarrow$

$\Rightarrow \sum_{n=1}^{+\infty} |n^2 3^n (x-2)^n|$ converges

$\Rightarrow \sum_{n=1}^{+\infty} n^2 3^n (x-2)^n$ converges.

Case 2 : Assume that $x \in (-\infty, 5/3) \cup (7/3, +\infty)$

Then: $3|x-2|>1 \Rightarrow \lim |a_n|^{1/n} > 1 \Rightarrow$

$\Rightarrow \sum_{n=1}^{+\infty} n^2 3^n (x-2)^n$ diverges.

Case 3 : Assume that $x = 7/3$. Then

$\forall n \in \mathbb{N}^*: a_n = n^2 3^n (x-2)^n = n^2 3^n (7/3 - 6/3)^n = n^2 3^n (1/3)^n$

$= n^2$.

Since $\lim_{x\to+\infty} x^2 = +\infty \Rightarrow a_n$ diverges $\Rightarrow \lim a_n \neq 0 \Rightarrow$

$\Rightarrow \sum_{n=1}^{+\infty} n^2 3^n (x-2)^n$ diverges.

Case 4 : Assume that $x = 5/3$

Then:

$\forall n \in \mathbb{N}^*: a_n = n^2 3^n (x-2)^n = n^2 3^n (5/3 - 2)^n = n^2 3^n (-1/3)^n$

$= n^2 (-1)^n$.

$\Rightarrow \forall n \in \mathbb{N}^*: (a_{2n} = (2n)^2 \wedge a_{2n+1} = -(2n+1)^2)$

Since

$\lim_{t\to+\infty}(4t^2) = +\infty \Rightarrow a_{4n}$ diverges $\Rightarrow a_n$ diverges $\Rightarrow$

$\Rightarrow \lim a_n \neq 0 \Rightarrow \sum_{n=1}^{+\infty} n^2 3^n (x-2)^n$ diverges

It follows that $A = (513, 713)$.

Uniform convergence of power series

Motivation

Let $f_n : A \to \mathbb{R}$ be a sequence of functions with $n \in \mathbb{N}^*$ such that $(\forall x \in A : f_n(x)$ converges$)$. We may thus define a new function $f : A \to \mathbb{R}$ as:

$$\forall x \in A : f(x) = \lim_{n \in \mathbb{N}^*} f_n(x)$$

The questions that we wish to address are the following:

a) If $(\forall n \in \mathbb{N}^* : f_n$ continuous on $A)$ can we conclude that $(f$ continuous on $A)$?

b) If $(\forall n \in \mathbb{N}^* : f_n$ integrable on $A)$ and $I \subseteq A$ is an interval, does it follow that

$$\lim_{n \in \mathbb{N}^*} \int_I f_n(x)\,dx = \int_I \left[\lim_{n \in \mathbb{N}^*} f_n(x)\right] dx \quad ??$$

c) If $(\forall n \in \mathbb{N}^* : f_n$ differentiable on $A)$, does it follow that

$$\frac{d}{dx} \lim_{n \in \mathbb{N}^*} f_n(x) = \lim_{n \in \mathbb{N}^*} (d/dx) f_n(x) \quad ??$$

The answer to the above questions is NO, in general, as we will illustrate with counterexamples. This will lead us to define the concept of a uniform convergent series.

The statements (a) and (b) will be shown to be true if the sequence of functions f_n converges uniformly. (c) holds for power series, but uniform convergence is not sufficient.

COUNTEREXAMPLES

a) Consider the function sequence

$\forall n \in \mathbb{N}^*: \forall x \in [0,1]: f_n(x) = x^n$.

Obviously: $\forall n \in \mathbb{N}^*$: f_n continuous on $[0,1]$.

However:

$$\forall x \in [0,1]: f(x) = \lim_{n \in \mathbb{N}^*} f_n(x) = \lim_{n \in \mathbb{N}^*} x^n = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in [0,1) \end{cases}$$

which is obviously not continuous at $x = 1$.

So in general, continuity is not always transmitted from f_n to f.

b) Consider the function sequence

$\forall n \in \mathbb{N}^*: \forall x \in [0,1]: f_n(x) = nx(1-x^2)^n$

We have:

$$\int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x^2)^n\,dx = \frac{-n}{2}\int_0^1 (-2x)(1-x^2)^n\,dx =$$

$$= \frac{-n}{2}\int_0^1 (1-x^2)^n (1-x^2)'\,dx =$$

$$= \frac{-n}{2(n+1)}\int_0^1 (n+1)(1-x^2)^n (1-x^2)'\,dx$$

$$= \frac{-n}{2(n+1)}\int_0^1 \left[(1-x^2)^{n+1}\right]'\,dx =$$

$$= \frac{-n}{2(n+1)} \left[(1-x^2)^{n+1} \right]_0^1 = \frac{-n}{2(n+1)} \left[(1-1^2)^{n+1} - (1-0)^{n+1} \right]$$

$$= \frac{n}{2(n+1)}, \quad \forall n \in \mathbb{N}^*$$

Since:

$$\lim_{t \to +\infty} \frac{t}{2(t+1)} = \lim_{t \to +\infty} \frac{t}{2t} = \frac{1}{2} \Rightarrow$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} \int_0^1 f_n(x)\,dx = \lim_{n \in \mathbb{N}^*} \frac{n}{2(n+1)} = \boxed{\frac{1}{2}} \quad (1)$$

However, let $x \in (0,1)$ be given and note that

$$0 < x < 1 \Rightarrow 0 < x^2 < 1 \Rightarrow 0 > -x^2 > -1 \Rightarrow -1 < -x^2 < 0$$
$$\Rightarrow 0 < 1 - x^2 < 1.$$

and also that

$$\forall n \in \mathbb{N}^*: \frac{f_{n+1}(x)}{f_n(x)} = \frac{(n+1)x(1-x^2)^{n+1}}{nx(1-x^2)^n} = \frac{(n+1)(1-x^2)}{n}$$

and therefore, since:

$$\lim_{t \to +\infty} \frac{t+1}{t} = \lim_{t \to +\infty} \frac{t}{t} = 1 \Rightarrow \lim_{n \in \mathbb{N}^*} \frac{n+1}{n} = 1 \Rightarrow$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} \frac{f_{n+1}(x)}{f_n(x)} = \lim_{n \in \mathbb{N}^*} \frac{(n+1)(1-x^2)}{n} = (1-x^2) \lim_{n \in \mathbb{N}^*} \frac{n+1}{n}$$

$$= 1 - x^2 < 1 \Rightarrow \quad (2)$$

We also note that $\forall n \in \mathbb{N}^*: \forall x \in (0,1): f_n(x) > 0 \quad (3)$

From Eq.(2) and Eq.(3) it follows that

$$\left(\forall x \in (0,1): \lim_{n \in \mathbb{N}^*} f_n(x) = 0 \right) \Rightarrow \int_0^1 \left[\lim_{n \in \mathbb{N}^*} f_n(x) \right] dx = \boxed{0.} \quad (4)$$

From Eq. (1) and Eq. (4) we have:

$$\begin{cases} \lim\limits_{n\in\mathbb{N}^*} \int_0^1 f_n(x)\,dx = \dfrac{1}{2} \\ \int_0^1 \left[\lim\limits_{n\in\mathbb{N}^*} f_n(x)\right] dx = 0 \end{cases} \Rightarrow \lim_{n\in\mathbb{N}^*} \int_0^1 f_n(x)\,dx \neq \int_0^1 \left[\lim_{n\in\mathbb{N}^*} f_n(x)\right] dx$$

We see, as a result, that the integral and the limit cannot be interchanged.

Uniform convergence definition

Let $f_n : A \to \mathbb{R}$ be a sequence of functions with $n \in \mathbb{N}^*$ and let $f : A \to \mathbb{R}$ be a function. We distinguish between pointwise convergence and uniform convergence as follows:

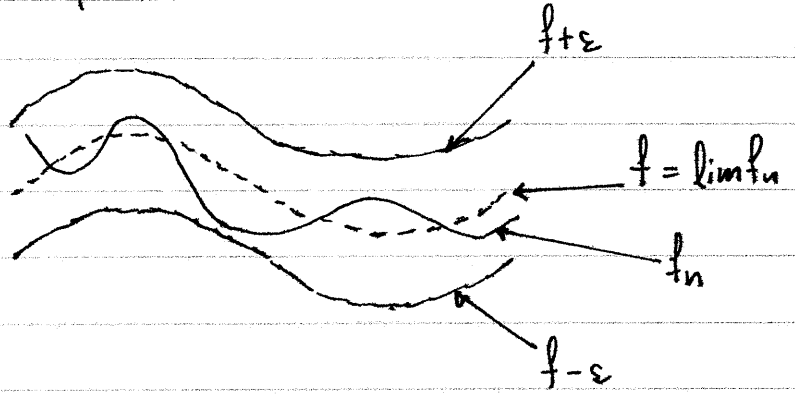
- f_n converges pointwise to f on $A \iff \forall x \in A : \lim_{n \in \mathbb{N}^*} f_n(x) = f(x) \iff$
  $\iff \forall x \in A : \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |f_n(x) - f(x)| < \varepsilon$

  Note that at the second step we use the definition of the sequence limit.

- f_n converges uniformly to f on $A \iff$
  $\iff \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \forall x \in A : |f_n(x) - f(x)| < \varepsilon$

↳ Note that the only difference between the two definitions is in the location of the $\forall x \in A$ quantifier. Note that similar quantifiers can be swaped (e.g. $\forall x \in A : \forall \varepsilon > 0$ vs. $\forall \varepsilon > 0 : \forall x \in A$) but we cannot swap dissimilar quantifiers (e.g. $\forall x \in A : \exists n_0 \in \mathbb{N}^*$ vs $\exists n_0 \in \mathbb{N}^* : \forall x \in A$) So the two definitions are logically different.

▶ Interpretation:



For any choice $\varepsilon > 0$, no matter how small, we can find a natural number $n_0 \in \mathbb{N}^*$ such that for all natural numbers $n > n_0$, the entire graph of $f_n(x)$ lies within a distance less than ε from the graph of $f(x)$. In other words, all points of the graph of $f_n(x)$ converge at the same rate towards the graph of $f(x)$. Pointwise convergence only requires each $f_n(x)$ to converge to $f(x)$ eventually, but for some x the convergence can be very fast and for other x excruciatingly slow.

Consequences of Uniform Convergence

(1) Preservation of continuity

Thm:

$$\begin{cases} f_n \text{ converges uniformly to } f \text{ on } A \Rightarrow f \text{ continuous on } x_0 \\ \forall n \in \mathbb{N}^*: f_n \text{ continuous on } x_0 \in A \end{cases}$$

Proof

It is sufficient to show that:

$$\forall \varepsilon > 0: \exists \delta > 0: \forall x \in A: (0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon)$$

Let $\varepsilon > 0$ be given. We need to define an appropriate $\delta > 0$.

Since:

f_n converges uniformly to f on $A \Rightarrow$

$$\Rightarrow \forall \varepsilon_0 > 0: \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: \forall p \in A: |f_n(p) - f(p)| < \varepsilon_0$$

We choose $\varepsilon_0 = \varepsilon/3$ and fix $n_0 \in \mathbb{N}^*$ such that

$\forall n \in \mathbb{N}^* - [n_0] : \forall p \in A : |f_n(p) - f(p)| < \varepsilon/3$

We define, for convenience, $N = n_0 + 1$. Then it follows that for $n = N$, we have: $\forall p \in A : |f_N(p) - f(p)| < \varepsilon/3$. (1)

Since f_N continuous on $x_0 \Rightarrow \lim_{x \to x_0} f_N(x) = f_N(x_0) \Rightarrow$

$\Rightarrow \forall \varepsilon_0 > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x - x_0| < \delta \Rightarrow |f_N(x) - f_N(x_0)| < \varepsilon_0)$

Choose $\varepsilon_0 = \varepsilon/3$ and fix $\delta > 0$ such that

$\forall x \in A : (0 < |x - x_0| < \delta \Rightarrow |f_N(x) - f_N(x_0)| < \varepsilon/3)$ (2)

Let $x \in A$ be given and assume that $0 < |x - x_0| < \delta$. Then, it follows, from Eq.(2), that

$|f_N(x) - f_N(x_0)| < \varepsilon/3$ (3)

Furthermore, via Eq.(1),

for $p = x$: $|f_N(x) - f(x)| < \varepsilon/3$ (4)

for $p = x_0$: $|f_N(x_0) - f(x_0)| < \varepsilon/3$ (5)

From Eq.(3), Eq.(4), and Eq.(5):

$|f(x) - f(x_0)| = |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \leq$

$\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)|$

$\leq |f_N(x) - f(x)| + \varepsilon/3 + \varepsilon/3 < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon.$

From the above, it follows that

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon)$

$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0) \Rightarrow f$ continuous on x_0. $\square$

② Uniform convergence and integration

Thm : We assume that

$$\begin{cases} f_n \text{ converges uniformly to } f \text{ on } [a,b] \\ \forall n \in \mathbb{N}^* : f_n \text{ continuous on } [a,b] \\ \forall n \in \mathbb{N}^* : \forall x \in [a,b] : g_n(x) = \int_a^x f_n(t)\,dt \\ \forall x \in A : g(x) = \int_a^x f(t)\,dt \end{cases}$$

Then: g_n converges uniformly to g on $[a,b]$.

↳ This theorem implies that under the above assumptions, we can interchange the sequence limit and integral as follows:

$$\forall x \in [a,b] : \lim_{n \in \mathbb{N}^*} \int_a^x f_n(t)\,dt = \int_a^x \left[\lim_{n \in \mathbb{N}^*} f_n(t) \right] dt$$

Proof

It is sufficient to show that

$$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \forall p \in [a,b] : |g_n(p) - g(p)| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since

f_n converges uniformly to f on $[a,b] \Rightarrow$

$$\Rightarrow \forall \varepsilon_0 > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \forall t \in [a,b] : |f_n(t) - f(t)| < \varepsilon_0$$

Choose $\varepsilon_0 = \varepsilon/(b-a)$ and fix $n_0 \in \mathbb{N}^*$ such that

$$\forall n \in \mathbb{N}^* - [n_0] : \forall t \in [a,b] : |f_n(t) - f(t)| < \frac{\varepsilon}{b-a}$$

Let $n \in \mathbb{N}^* - [n_0]$ and $p \in [a,b]$ be given. Then:

$$|g_n(p)-g(p)| = \left| \int_a^p f_n(t)\,dt - \int_a^p f(t)\,dt \right| = \left| \int_a^p [f_n(t)-f(t)]\,dt \right|$$

$$\leq \int_a^p |f_n(t)-f(t)|\,dt \leq \int_a^b |f_n(t)-f(t)|\,dt <$$

$$< \int_a^b \frac{\varepsilon}{b-a}\,dt = \frac{\varepsilon}{b-a}\int_a^b dt = \frac{\varepsilon}{b-a}\,[t]_a^b$$

$$= \frac{\varepsilon}{b-a}(b-a) = \varepsilon \Rightarrow \underline{|g_n(p)-g(p)| < \varepsilon}$$

From the above argument it follows that

$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \forall p \in [a,b] : |g_n(p)-g(p)| < \varepsilon$

$\Rightarrow g_n$ converges uniformly to g on $[a,b]$.

Sufficient condition for uniform convergence

Thm: Let $f_n: A \to \mathbb{R}$ be a sequence of functions. Then

$$\begin{cases} \forall n \in \mathbb{N}^*: \forall x \in S: 0 \leq |f_n(x)| \leq M_n \Rightarrow \sum_{n=1}^{+\infty} f_n(x) \text{ converges uniformly} \\ \sum_{n=1}^{+\infty} M_n \text{ converges} \end{cases} \quad \text{on } S$$

Proof

First, we note that from the comparison test:

$$\begin{cases} \forall n \in \mathbb{N}^*: \forall x \in S: 0 \leq |f_n(x)| \leq M_n \Rightarrow \sum_{n=1}^{+\infty} |f_n(x)| \text{ converges} \quad (1) \\ \sum_{n=1}^{+\infty} M_n \text{ converges} \end{cases}$$

Sufficient to show that

$$\forall \varepsilon > 0: \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: \forall x \in S: \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{+\infty} f_k(x) \right| < \varepsilon$$

Let $\varepsilon > 0$ be given. Since

$\sum_{n=1}^{+\infty} M_n$ converges $\Rightarrow$

$$\Rightarrow \forall \varepsilon_0 > 0: \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: \left| \sum_{k=1}^{n} M_k - \sum_{k=1}^{+\infty} M_k \right| < \varepsilon_0$$

Choose $\varepsilon_0 = \varepsilon$ and fix $n_0 \in \mathbb{N}^*$ such that

$$\forall n \in \mathbb{N}^* - [n_0]: \left| \sum_{k=n+1}^{+\infty} M_k \right| < \varepsilon$$

Let $n \in \mathbb{N}^* - [n_0]$ and let $x \in S$ be given. Then:

$$\left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{+\infty} f_k(x) \right| = \left| \sum_{k=n+1}^{+\infty} f_k(x) \right| \leq \sum_{k=n+1}^{+\infty} |f_k(x)| \leq$$

$$\leq \sum_{k=n+1}^{+\infty} |f_k(x)| \leq \sum_{k=n+1}^{+\infty} M_k \leq \left| \sum_{k=n+1}^{+\infty} M_k \right| < \varepsilon \Rightarrow$$

$$\Rightarrow \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{+\infty} f_k(x) \right| < \varepsilon$$

From the above argument it follows that

$$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \forall x \in S : \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{+\infty} f_k(x) \right| < \varepsilon$$

$$\Rightarrow \sum_{k=1}^{+\infty} f_k(x) \text{ converges uniformly on } S. \quad \square$$

↳ This result is known as the Weierstrass M-test.

Application to series

Given a series $\sum_{n=1}^{+\infty} f_n(x)$ that converges uniformly on $[a,b]$ we may immediately conclude that

$$(\forall n \in \mathbb{N}^*: f_n \text{ continuous on } [a,b]) \Rightarrow \sum_{n=1}^{+\infty} f_n(x) \text{ continuous on } [a,b]$$

$$\forall t \in [a,b]: \int_a^t \left[\sum_{n=1}^{+\infty} f_n(x) \right] dx = \sum_{n=1}^{+\infty} \int_a^t f_n(x)\,dx$$

Unfortunately, uniform convergence is not strong enough to allow a similar result for differentiability such as:

$$\forall x \in [a,b]: \frac{d}{dx} \sum_{k=1}^{+\infty} f_k(x) = \sum_{k=1}^{+\infty} \frac{d}{dx} [f_k(x)] \leftarrow \text{NOT ALWAYS TRUE!!}$$

For example, for $\forall n \in \mathbb{N}^*: \forall x \in \mathbb{R}: f_n(x) = \frac{\sin(nx)}{n^2}$ we will show that

a) $\sum_{n=1}^{+\infty} f_n(x)$ converges uniformly on $\mathbb{R}$; but

b) $\sum_{n=1}^{+\infty} f_n'(x)$ diverges for $x=0$.

Proof

We note that since:

$$\forall x \in \mathbb{R}: |f_n(x)| = \left| \frac{\sin(nx)}{n^2} \right| = \frac{|\sin(nx)|}{n^2} \leq \frac{1}{n^2}$$

we have

$$\begin{cases} \forall x \in \mathbb{R}: 0 \leq |f_n(x)| \leq 1/n^2 \\ \sum_{n=1}^{+\infty} (1/n^2) \text{ converges} \end{cases} \Rightarrow \sum_{n=1}^{+\infty} f_n(x) \text{ converges uniformly on } \mathbb{R}.$$

On the other hand:

$$\sum_{n=1}^{+\infty} f_n'(x) = \sum_{n=1}^{+\infty} \left[\frac{d}{dx} \left(\frac{\sin(nx)}{n^2} \right) \right] = \sum_{n=1}^{+\infty} \frac{n\cos(nx)}{n^2} =$$

$$= \sum_{n=1}^{+\infty} \frac{\cos(nx)}{n} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} f_n'(0) = \sum_{n=1}^{+\infty} \frac{\cos 0}{n} = \sum_{n=1}^{+\infty} \frac{1}{n} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} f_n'(0) \text{ diverges.} \qquad \square$$

↳ We see that term-by-term differentiation of a series will not always work, even if the original series expansion converges uniformly.

Properties of power series.

We consider a function $f(x)$ defined via a power-series of the form

$$f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

We wish to determine:

1) The domain A of $f(x)$: for which $x \in A$ does the power series converge.
2) The integral of $f(x)$.
3) The derivative of $f(x)$, if it exists.

Radius of convergence

Thm:

a) $\sum_{n=0}^{+\infty} a_n x^n$ converges for $x = b$ $\Rightarrow$ $\sum_{n=0}^{+\infty} a_n x^n$ converges uniformly on $(-|b|, |b|)$

b) $\sum_{n=0}^{+\infty} a_n x^n$ diverges for $x = b$ $\Rightarrow$ $\forall x \in (-\infty, -|b|) \cup (|b|, +\infty)$: $\sum_{n=1}^{+\infty} a_n x^n$ diverges

Proof

a) Let $x \in (-|b|, |b|)$ be given. Since:

$\sum_{n=0}^{+\infty} a_n b^n$ converges $\Rightarrow \lim_{n \in \mathbb{N}^*} a_n b^n = 0 \Rightarrow$

$\Rightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n b^n| < \varepsilon$

Choose $\varepsilon = 1$ and fix $n_0 \in \mathbb{N}^*$ such that
$\forall n \in \mathbb{N}^* - [n_0] : |a_n b^n| < 1$
Let $n \in \mathbb{N}^* - [n_0]$ be given. Then:

$$|a_n x^n| = \left| \frac{a_n b^n x^n}{b^n} \right| = |a_n b^n| \cdot \left| \frac{x^n}{b^n} \right| < \left| \frac{x^n}{b^n} \right| =$$

$$= \left| \left(\frac{x}{b} \right)^n \right| = \left| \frac{x}{b} \right|^n$$

It follows that

$$\forall n \in \mathbb{N}^* - [n_0] : \forall x \in (-b, b) : 0 \leqslant |a_n x^n| < \left| \frac{x}{b} \right|^n \qquad (1)$$

We also know that

$$\forall x \in (-b, b) : \sum_{n=0}^{+\infty} \left| \frac{x}{b} \right|^n \text{ converges} \qquad (2)$$

From Eq.(1) and Eq.(2), it follows that
$\sum_{n=0}^{+\infty} a_n x^n$ converges uniformly on $(-|b|, |b|)$.

b) Assume that $\sum_{n=1}^{+\infty} a_n b^n$ diverges. Let $x \in (-\infty, -|b|) \cup (|b|, +\infty)$ be given. To show that $\sum_{n=1}^{+\infty} a_n x^n$ diverges, we use proof by contradiction and assume that

$$\sum_{n=1}^{+\infty} a_n x^n \text{ converges} \qquad (1)$$

Since:

$x \in (-\infty, -|b|) \cup (|b|, +\infty) \Rightarrow |x| > |b| \Rightarrow |b| < |x| \Rightarrow b \in (-|x|, |x|) \quad (2)$

then, from Eq.(1) and Eq.(2) it follows, via the previous result, that

$\sum_{n=1}^{+\infty} a_n b^n$ converges

which is a contradiction, since it is assumed that $\sum_{n=1}^{+\infty} a_n b^n$ diverges. It follows that $\sum_{n=1}^{+\infty} a_n x^n$ diverges □.

↳ Shifting the series around x_0 gives the following generalization:

$\sum_{n=0}^{+\infty} a_n (x-x_0)^n$ converges on $x = x_0 + b \Rightarrow$

$\Rightarrow \sum_{n=1}^{+\infty} a_n (x-x_0)^n$ converges uniformly on $(x_0-|b|, x_0+|b|)$

↳ Consequences of uniform convergence

- From the above, it immediately follows that given a function $f(x)$ defined as:

$$f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

If $f(x)$ converges at $x=b$, then $f(x)$ converges uniformly on $(x_0-|b|, x_0+|b|)$, and we immediately conclude that

a) f is continuous on $(x_0-|b|, x_0+|b|)$

b) $\forall c_1, c_2 \in (x_0-|b|, x_0+|b|)$: $\int_{c_1}^{c_2} f(x)\,dx = \int_{c_1}^{c_2} \left[\sum_{n=0}^{+\infty} a_n (x-x_0)^n\right] dx$

$$= \sum_{n=0}^{+\infty} \left[\int_{c_1}^{c_2} a_n (x-x_0)^n dx\right] = \sum_{n=0}^{+\infty} \left[\frac{a_n (x-x_0)^{n+1}}{n+1}\right]_{x=c_1}^{x=c_2} =$$

$$= \sum_{n=0}^{+\infty} \left[\frac{a_n (c_2 - x_0)^{n+1} - a_n (c_1 - x_0)^{n+1}}{n+1} \right]$$

- From the proof of the radius of convergence theorem, it is obvious that if we use the comparison test instead of the Weierstrass M-test, we also get the statement:

$$\sum_{n=1}^{+\infty} a_n (x - x_0)^n \text{ Converges on } X = x_0 + b$$

$$\Rightarrow \quad \forall x \in (x_0 - |b|, x_0 + |b|) : \sum_{n=1}^{+\infty} \left| a_n (x - x_0)^n \right| \text{ converges}$$

- Another immediate consequence of the uniform convergence of a power-series defining a function is that we can integrate term-by-term with an indefinite integral and find the antiderivative of the function:

$$\int f(x)\,dx = \int \left[\sum_{n=0}^{+\infty} a_n (x - x_0)^n \right] dx = \sum_{n=0}^{+\infty} \left[\int a_n (x - x_0)^n \, dx \right] =$$

$$= \sum_{n=0}^{+\infty} \frac{a_n (x - x_0)^{n+1}}{n+1} + C$$

- As we explained before, uniform convergence is not sufficient to justify term-by-term differentiation. However, we will now show that power series can be always differentiated.

Differentiation of a power series

The following theorem allows the term-by-term differentiation of power series

Thm: We define the function

$$\forall x \in [x_0-b, x_0+b] : f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

assuming the corresponding series converges on (x_0-b, x_0+b).
Then:

a) f differentiable on (x_0-b, x_0+b)

b) $\forall x \in (x_0-b, x_0+b) : f'(x) = \sum_{n=1}^{+\infty} n a_n (x-x_0)^{n-1}$

Proof

Let $x \in (x_0-b, x_0+b)$ be given. Since for $x = x_0+b$:

$f(x_0+b) = \sum_{n=1}^{+\infty} a_n b^n$ converges $\Rightarrow \lim_{n \in \mathbb{N}^*} a_n b^n = 0 \Rightarrow$

$\Rightarrow a_n b^n$ bounded $\Rightarrow$

$\Rightarrow a_n b^n$ absolutely bounded

$\Rightarrow \exists M \in (0, +\infty) : \forall n \in \mathbb{N}^* : |a_n b^n| \leq M$

Fix an $M \in (0, +\infty)$ such that : $\forall n \in \mathbb{N}^* : |a_n b^n| \leq M$

Let $n \in \mathbb{N}^*$ be given. Then we have:

$$|n a_n (x-x_0)^{n-1}| = |n a_n| |x-x_0|^{n-1} = \frac{n |a_n| \cdot |x-x_0|^n |b^n|}{|x-x_0| \cdot |b^n|} =$$

$$= \frac{n}{|x-x_0|} \frac{|x-x_0|^n}{|b|^n} |a_n b^n| \leq \frac{nM}{|x-x_0|} \left| \frac{x-x_0}{b} \right|^n$$

It follows that

$$\forall n\in\mathbb{N}^*: |n a_n (x-x_0)^{n-1}| \leq \frac{nM}{|x-x_0|}\left|\frac{x-x_0}{b}\right|^n = \gamma_n \quad (1)$$

To show that $\sum_{n=1}^{+\infty}\gamma_n$ converges, we note that

$$\forall n\in\mathbb{N}^*: \frac{\gamma_{n+1}}{\gamma_n} = \frac{\frac{(n+1)M}{|x-x_0|}\left|\frac{x-x_0}{b}\right|^{n+1}}{\frac{nM}{|x-x_0|}\left|\frac{x-x_0}{b}\right|^{n}} = \frac{n+1}{n}\left|\frac{x-x_0}{b}\right| > 0 \quad (2)$$

and therefore:

$$\lim_{t\to+\infty}\frac{t+1}{t} = \lim_{t\to+\infty}\frac{t}{t} = 1 \Rightarrow \lim_{n\in\mathbb{N}^*}\frac{n+1}{n} = 1 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*}\frac{\gamma_{n+1}}{\gamma_n} = \lim_{n\in\mathbb{N}^*}\left[\frac{n+1}{n}\left|\frac{x-x_0}{b}\right|\right] = \left|\frac{x-x_0}{b}\right|\lim_{n\in\mathbb{N}^*}\frac{n+1}{n}$$

$$= \left|\frac{x-x_0}{b}\right|$$

Since $x\in(x_0-b, x_0+b) \Rightarrow x_0-b<x<x_0+b \Rightarrow -b<x-x_0<b$

$$\Rightarrow |x-x_0|<b=|b| \Rightarrow \frac{|x-x_0|}{|b|}<1 \Rightarrow \left|\frac{x-x_0}{b}\right|<1$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*}\frac{\gamma_{n+1}}{\gamma_n} < 1 \quad (3)$$

From the ratio test, via Eq. (2) and Eq. (3), we conclude that:

$$\sum_{n=1}^{+\infty}\gamma_n \text{ converges} \quad (4)$$

From Eq. (1) and Eq. (4), via the Weierstrass M-test, it follows

that $\sum_{n=1}^{+\infty} n a_n (x-x_0)^{n-1}$ converges uniformly on $x \in (x_0-b, x_0+b)$.

Define $\forall x \in (x_0-b, x_0+b)$: $g(x) = \sum_{n=1}^{+\infty} n a_n (x-x_0)^{n-1}$

and note that:

$$\forall x \in (x_0-b, x_0+b): \int_{x_0}^{x} g(t)\,dt = \int_{x_0}^{x} \left[\sum_{n=1}^{+\infty} n a_n (t-x_0)^{n-1} \right] dt =$$

$$= \sum_{n=1}^{+\infty} \left[n a_n \int_{x_0}^{x} (t-x_0)^{n-1}\,dt \right] =$$

$$= \sum_{n=1}^{+\infty} \left\{ n a_n \left[\frac{(t-x_0)^n}{n} \right]_{t=x_0}^{t=x} \right\} =$$

$$= \sum_{n=1}^{+\infty} \left\{ a_n \left[(x-x_0)^n - (x_0-x_0)^n \right] \right\} = \sum_{n=1}^{+\infty} a_n (x-x_0)^n = f(x)$$

From the fundamental theorem of calculus it follows that; f is differentiable on (x_0-b, x_0+b) and

$$f'(x) = \frac{d}{dx} \int_{x_0}^{x} g(t)\,dt = g(x) = \sum_{n=1}^{+\infty} n a_n (x-x_0)^{n-1}, \forall x \in (x_0-b, x_0+b).$$

□

Comments on power series differentiation

Consider a function $f(x)$ defined by a convergent power series over a closed interval $[x_0-b, x_0+b]$:

$$\forall x \in [x_0-b, x_0+b]: \quad f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

- Note that the differentiation theorem is only able to give us differentiability in the open interval (x_0-b, x_0+b):

$$\forall x \in (x_0-b, x_0+b): \quad f'(x) = \sum_{n=1}^{+\infty} n a_n (x-x_0)^{n-1}$$

- Because the theorem also gives uniform convergence of the power series of $f'(x)$, we can differentiate $f'(x)$ again term-by-term to obtain the following power series for the second derivative $f''(x)$.

$$\forall x \in (x_0-b, x_0+b): \quad f''(x) = \sum_{n=2}^{+\infty} n(n-1) a_n (x-x_0)^{n-2}$$

- Using proof by induction, the k^{th} derivative $f^{(k)}(x)$ can be shown to be given by the following power series:

$$\forall k \in \mathbb{N}^*: \forall x \in (x_0-b, x_0+b): \quad f^{(k)}(x) = \sum_{n=k}^{+\infty} a_n \left[\prod_{l=0}^{k-1} (n-l) \right] (x-x_0)^{n-k}$$

$$= \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} a_n (x-x_0)^{n-k}$$

- From the above expression for $f^{(k)}(x)$, using $x = x_0$ we find:

$$f^{(k)}(x_0) = \frac{k!}{(k-k)!}\, a_k = \frac{k!}{0!}\, a_k = k!\, a_k \Rightarrow a_k = \frac{f^{(k)}(x_0)}{k!}$$

It follows that:

$$\left[\forall x \in (x_0 - b, x_0 + b) : f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n\right] \Rightarrow \left(\forall n \in \mathbb{N} : a_n = \frac{f^{(n)}(x_0)}{n!}\right)$$

Because the immediate consequence of this result is that the coefficients a_n are <u>unique</u> (if they <u>exist</u>) it immediately follows that

$$\left[\exists c \in [0, +\infty) : \forall x \in (x_0 - c, x_0 + c) : \sum_{n=0}^{+\infty} a_n (x - x_0)^n = \sum_{n=0}^{+\infty} b_n (x - x_0)^n\right] \Leftrightarrow$$

$$\Leftrightarrow (\forall n \in \mathbb{N} : a_n = b_n)$$

This result in turn can be applied in

a) Solution of differential equations via series expansion

b) The proof of the binomial series.

Taylor expansions

- Let $f: A \to \mathbb{R}$ be a function, let $x_0 \in A$ and assume that $\forall n \in \mathbb{N}^*$: f is n-differentiable on A.
We define the n^{th}-order Taylor polynomial $T_n(x_0)f$ around x_0 as:

$$\forall x \in \mathbb{R}: (T_n(x_0)f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

- We have already shown that IF $f(x)$ can be represented by a power series around x_0 on an interval (x_0-b, x_0+b) then the Taylor polynomials $T_n(x_0)f$ will converge uniformly to the function $f(x)$ on (x_0-b, x_0+b), and therefore

$$\forall x \in (x_0-b, x_0+b): f(x) = \lim_{n \in \mathbb{N}^*} (T_n(x_0)f)(x)$$
$$= \sum_{k=0}^{+\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

- However, the converse is not true as:
(a) It is possible for the Taylor series to not converge
(b) OR, to converge but not converge to $f(x)$.

COUNTEREXAMPLE

Consider the function

$$f(x) = \begin{cases} \exp(-1/x^2) & , \text{if } x \in \mathbb{R} - \{0\} \\ 0 & , \text{if } x = 0 \end{cases}$$

Via the limit definition of the derivative and using a recurrence argument we can show that

$$\forall n \in \mathbb{N}^* : f^{(n)}(0) = 0$$

(note: proving this is very challenging, so it is left for you as homework).

It follows that the Taylor series converges on $\mathbb{R}$ as follows:

$$\forall x \in \mathbb{R} : \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$$

However, we see that, except for $x = 0$, the series does not converge to $f(x)$ for $x \in \mathbb{R} - \{0\}$.

↳ To show that a Taylor series converges and converges to the original function $f(x)$, there are three methods:

1) Term-by-term differentiation or integration of a previously established Taylor expansion
2) Directly, via convergence theorems
3) Via special techniques (e.g. the Binomial series).

(I) → Term-by-term differentiation or integration

We use the geometric series as a point of departure:

$$\forall x \in (-1,1): \frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$$

to establish the following results:

$$\forall x \in (-1,1): \operatorname{Arctan}(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\forall x \in (-1,1): \ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

Proof

a) Let $x \in (-1,1)$ be given. Then:

$$\operatorname{Arctan}(x) = \int_0^x \frac{dt}{1+t^2} = \int_0^x \frac{dt}{1-(-t^2)} \overset{x\in(-1,1)}{=}$$

$$= \int_0^x \left[\sum_{n=0}^{+\infty} (-t^2)^n\right] dt = \sum_{n=0}^{+\infty} \left[\int_0^x (-t^2)^n dt\right] =$$

$$= \sum_{n=0}^{+\infty} \left[(-1)^n \int_0^x t^{2n} dt\right] = \sum_{n=0}^{+\infty} \left[(-1)^n \left[\frac{t^{2n+1}}{2n+1}\right]_0^x\right]$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

and therefore

$$\forall x \in (-1,1): \operatorname{Arctan}(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

b) Let $x \in (-1,1)$ be given. Then:

$$\ln(1+x) = \ln(1+x) - \ln 1 = \int_0^x [\ln(1+t)]' \, dt = \int_0^x \frac{(1+t)'}{1+t} \, dt$$

$$= \int_0^x \frac{dt}{1+t} = \int_0^x \frac{dt}{1-(-t)} \overset{x \in (-1,1)}{=} \int_0^x \left[\sum_{n=0}^{+\infty} (-t)^n \right] dt$$

$$= \sum_{n=0}^{+\infty} \left[\int_0^x (-t)^n \, dt \right] = \sum_{n=0}^{+\infty} \left[(-1)^n \int_0^x t^n \, dt \right]$$

$$= \sum_{n=0}^{+\infty} \left[(-1)^n \left[\frac{t^{n+1}}{n+1} \right]_0^x \right] = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1} - 0^{n+1}}{n+1}$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad \square$$

↳ These results have the following immediate applications:

1) Calculation of π:

$$\boxed{\pi = \sum_{n=0}^{+\infty} \frac{4(-1)^n}{2n+1}}$$

Proof

Since $\tan(\pi/4) = 1 \Rightarrow \pi/4 = \text{Arctan}(1) \Rightarrow$

$$\Rightarrow \pi = 4\,\text{Arctan}(1) = 4 \sum_{n=0}^{+\infty} (-1)^n \frac{1^{2n+1}}{2n+1} = 4 \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} =$$

$$= \sum_{n=0}^{+\infty} \frac{4(-1)^n}{2n+1}$$

2) Calculation of logarithm

$$\forall x \in (0,2): \ln x = \sum_{n=0}^{+\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

$$\forall x \in [2,+\infty): \ln x = \sum_{n=0}^{+\infty} \frac{1}{n+1} \left(\frac{x-1}{x}\right)^{n+1}$$

Proof

The first equation is obtained from the Taylor expansion of $\ln(1+x)$ around $x_0 = 0$, by shifting the variable x.

For the second equation, let $x \in [2,+\infty)$ be given. Then

$$\ln x = -\ln\left(\frac{1}{x}\right) = -\ln\left(1+\frac{1}{x}-1\right) = -\ln\left(1+\frac{1-x}{x}\right) \quad (1)$$

and

$$x \in [2,+\infty) \Rightarrow x > 2 > 0 \Rightarrow 0 < \frac{1}{x} < \frac{1}{2} \Rightarrow$$

$$\Rightarrow -1 < \frac{1}{x} - 1 < \frac{1}{2} - 1 \Rightarrow -1 < \frac{1-x}{x} < \frac{-1}{2}$$

$$\Rightarrow \frac{1-x}{x} \in (-1,1) \quad (2)$$

From Eq.(1) and Eq.(2):

$$\ln x = -\ln\left(1+\frac{1-x}{x}\right) = -\sum_{n=0}^{+\infty} (-1)^n \frac{1}{n+1} \left(\frac{1-x}{x}\right)^{n+1} =$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{n+1} \left(\frac{1-x}{x}\right)^{n+1} = \sum_{n=0}^{+\infty} \frac{1}{n+1} \left(\frac{x-1}{x}\right)^{n+1}$$

EXAMPLES

↦ **Method** : Taylor expansions of rational functions can be evaluated via geometric series and partial function decomposition

a) Taylor expand $f(x) = \dfrac{10}{3x^2-14x+8}$ around $x=0$.

Solution

We factor $3x^2-14x+8$:

$\Delta = b^2-4ac = (-14)^2 - 4\cdot 3\cdot 8 = 196 - 12\cdot 8 = 196-96 = 100 = 10^2$

$\Rightarrow x_{1,2} = \dfrac{-b\pm\sqrt{\Delta}}{2a} = \dfrac{-(-14)\pm 10}{2\cdot 3} = \dfrac{14\pm 10}{6} =$

$= \begin{cases} 24/6 = 4 \\ 4/6 = 2/3 \end{cases}$

and therefore $3x^2-14x+8 = 3(x-4)(x-2/3)$. It follows that

$$f(x) = \frac{10}{3x^2-14x+8} = \frac{A}{x-4} + \frac{B}{x-2/3} = \frac{-A}{4-x} + \frac{-B}{2/3-x} =$$

$$= \frac{-A}{4}\,\frac{1}{1-(x/4)} + \frac{-B}{2/3}\,\frac{1}{1-(3x/2)} =$$

$$= \frac{-A}{4}\sum_{k=0}^{+\infty}\left(\frac{x}{4}\right)^k + \frac{-3B}{2}\sum_{k=0}^{+\infty}\left(\frac{3x}{2}\right)^k$$

$$= \sum_{k=0}^{+\infty}\left[\frac{-A}{4\cdot 4^k}x^k + \frac{-3B}{2}\left(\frac{3}{2}\right)^k x^k\right] =$$

$$= \sum_{k=0}^{+\infty} \left[\frac{-A}{4^{k+1}} + \frac{-3^{k+1} B}{2^{k+1}} \right] x^k$$

with

$$A = \left. \frac{10}{3(x-2/3)} \right|_{x=4} = \left. \frac{10}{3x-2} \right|_{x=4} = \frac{10}{3 \cdot 4 - 2} =$$

$$= \frac{10}{12-2} = \frac{10}{10} = 1$$

$$B = \left. \frac{10}{3(x-4)} \right|_{x=2/3} = \frac{10}{3(2/3-4)} = \frac{10}{2-12} =$$

$$= \frac{10}{-10} = -1$$

consequently,

$$f(x) = \sum_{k=0}^{+\infty} \left[\frac{-1}{4^{k+1}} + \frac{-3^{k+1}(-1)}{2^{k+1}} \right] x^k =$$

$$= \sum_{k=0}^{+\infty} \left[\frac{3^{k+1}}{2^{k+1}} - \frac{1}{4^{k+1}} \right] x^k$$

To determine the radius of convergence, we note that the two geometric expansions require

$$\begin{cases} |x/4| < 1 \\ |3x/2| < 1 \end{cases} \Leftrightarrow \begin{cases} |x|/4 < 1 \\ 3|x|/2 < 1 \end{cases} \Leftrightarrow \begin{cases} |x| < 4 \\ |x| < 2/3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow |x| < 2/3 \Leftrightarrow x \in (-2/3, 2/3)$$

It follows that

$$\forall x \in (-2/3, 2/3): f(x) = \frac{10}{3x^2 - 14x + 8} = \sum_{k=0}^{+\infty} \left[\frac{3^{k+1}}{2^{k+1}} - \frac{1}{4^{k+1}} \right] x^k$$

↦ Method: Repeating factors on the denominator can be addressed with term-by-term differentiation.

b) Taylor expand $f(x) = \dfrac{x^2}{x^2+4x+4}$ around $x=0$.

Solution

We note that

$$f(x) = \frac{x^2}{x^2+4x+4} = \frac{x^2}{(x+2)^2} = x^2(x+2)^{-2} =$$

$$= -x^2\left[(-1)(x+2)^{-2}\right] = -x^2\left[(x+2)^{-1}\right]' =$$

$$= -x^2\frac{d}{dx}\left[\frac{1}{x+2}\right] = \frac{-x^2}{2}\frac{d}{dx}\left[\frac{1}{1+x/2}\right] =$$

$$= \frac{-x^2}{2}\frac{d}{dx}\left[\frac{1}{1-(-x/2)}\right] = \frac{-x^2}{2}\frac{d}{dx}\left[\sum_{n=0}^{+\infty}\left(\frac{-x}{2}\right)^n\right]$$

$$= \frac{-x^2}{2}\sum_{n=0}^{+\infty}\frac{d}{dx}\left[\frac{(-1)^n}{2^n}x^n\right] = \frac{-x^2}{2}\sum_{n=1}^{+\infty}\frac{(-1)^n}{2^n}nx^{n-1}$$

$$= \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{2^{n+1}}nx^{n+1} = \sum_{n=2}^{+\infty}\frac{(-1)^n(n-1)}{2^n}x^n$$

To determine the radius of convergence, we require that

$|-x/2|<1 \Leftrightarrow (1/2)|x|<1 \Leftrightarrow |x|<2 \Leftrightarrow -2<x<2$

$\Leftrightarrow x\in(-2,2)$

and therefore

$$\forall x\in(-2,2): f(x) = \frac{x^2}{x^2+4x+4} = \sum_{n=2}^{+\infty}\frac{(-1)^n(n-1)}{2^n}x^n$$

(2) → Via convergence theorem

We can estimate the error, when truncating a Taylor expansion and establish convergence via the following theorems:

Thm: Let $f: A \to \mathbb{R}$ be a function with $(x_0-b, x_0+b) \subseteq A$

We assume that

$$\begin{cases} \forall k \in [n+1]: f^{(k)} \text{ differentiable on } (x_0-b, x_0+b) \\ f^{(n+1)} \text{ continuous on } (x_0-b, x_0+b) \end{cases}$$

Then:

$$\forall x \in (x_0-b, x_0+b): f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + E_n(f|x, x_0)$$

with $$E_n(f|x, x_0) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)\,dt$$

Proof

Let $x \in (x_0-b, x_0+b)$ be given. For $n=1$:

$$E_1(f|x, x_0) = f(x) - \sum_{k=0}^{1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k =$$

$$= f(x) - f(x_0) - f'(x_0)(x-x_0) =$$

$$= \int_{x_0}^{x} f'(t)\,dt - f'(x_0)\int_{x_0}^{x} dt =$$

$$= \int_{x_0}^{x} [f'(t) - f'(x_0)]\,dt = \int_{x_0}^{x} (t-x)'[f'(t) - f'(x_0)]\,dt$$

$$= \Big\{(t-x)[f'(t)-f'(x_0)]\Big\}_{x_0}^{x} - \int_{x_0}^{x} (t-x)[f'(t)-f'(x_0)]'\,dt$$

$$= (x-x)\left[f'(x)-f'(x_0)\right] - (x_0-x)\left[f'(x_0)-f'(x_0)\right] - \int_{x_0}^{x}(t-x)f''(t)\,dt$$

$$= \int_{x_0}^{x}(x-t)f''(t)\,dt$$

For $n=m$, we assume that

$$f(x) = \sum_{k=0}^{m}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + E_m(f|x,x_0)$$

with $E_m(f|x,x_0) = \frac{1}{m!}\int_{x_0}^{x}(x-t)^m f^{(m+1)}(t)\,dt$

For $n=m+1$, we have:

$$E_m(f|x,x_0) = \frac{1}{m!}\int_{x_0}^{x}(x-t)^m f^{(m+1)}(t)\,dt = \frac{1}{m!}\int_{x_0}^{x}\left[\frac{-(x-t)^{m+1}}{m+1}\right]' f^{(m+1)}(t)\,dt$$

$$= \frac{1}{m!}\left[\frac{-(x-t)^{m+1}}{m+1}f^{(m+1)}(t)\right]_{t=x_0}^{t=x} - \frac{1}{m!}\int_{x_0}^{x}\frac{-(x-t)^{m+1}}{m+1}\left[f^{(m+1)}(t)\right]'dt$$

$$= \frac{1}{m!}\left[\frac{-(x-x)^{m+1}}{m+1}f^{(m+1)}(x)\right] - \frac{1}{m!}\left[\frac{-(x-x_0)^{m+1}}{m+1}f^{(m+1)}(x_0)\right]$$

$$+ \frac{1}{(m+1)m!}\int_{x_0}^{x}(x-t)^{m+1}f^{(m+2)}(t)\,dt$$

$$= \frac{f^{(m+1)}(x_0)}{(m+1)m!}(x-x_0)^{m+1} + \frac{1}{(m+1)!}\int_{x_0}^{x}(x-t)^{m+1}f^{(m+2)}(t)\,dt$$

$$= \frac{f^{(m+1)}(x_0)}{(m+1)!}(x-x_0)^{m+1} + E_{m+1}(f|x,x_0)$$

and therefore

$$f(x) = \sum_{k=0}^{m}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + E_m(f|x,x_0) =$$

$$= \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(m+1)}(x_0)}{(m+1)!}(x-x_0)^{m+1} + E_{m+1}(f|x,x_0)$$

$$= \sum_{k=0}^{m+1} \frac{f^{(k)}(x_0)}{k!} + E_{m+1}(f|x,x_0)$$

It follows, by induction, that

$$\forall n \in \mathbb{N}^*: \forall x \in (x_0-\delta, x_0+\delta): f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + E_n(f|x,x_0) \quad \square$$

↔ The convergence of a Taylor series expansion can be established by the following theorem:

Thm: Let $f: A \to \mathbb{R}$ be a function with $(x_0-\delta, x_0+\delta) \subseteq A$ and assume that

$$\begin{cases} \forall n \in \mathbb{N}^*: f \text{ } n\text{-times differentiable on } (x_0-\delta, x_0+\delta) \\ \exists M \in (0,+\infty): \forall n \in \mathbb{N}^*: \forall x \in (x_0-\delta, x_0+\delta): |f^{(n)}(x)| \leq M^n \end{cases}$$

Then:

$$\forall x \in (x_0-\delta, x_0+\delta): f(x) = \sum_{n=1}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Proof

Let $x \in (x_0-\delta, x_0+\delta)$ be given and let $n \in \mathbb{N}^*$ be given. Without loss of generality, we assume that $x > x_0$. Then:

$$|E_n(x)| = \left| \frac{1}{n!} \int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)\,dt \right| \leq \frac{1}{n!} \int_{x_0}^{x} |(x-t)^n f^{(n+1)}(t)|\,dt$$

$$= \frac{1}{n!} \int_{x_0}^{x} |x-t|^n |f^{(n+1)}(t)|\,dt \leq \frac{1}{n!} \int_{x_0}^{x} |x-t|^n M^{n+1}\,dt$$

$$= \frac{M^{n+1}}{n!} \int_{x_0}^{x} (x-t)^n\,dt = \frac{M^{n+1}}{n!} \left[\frac{(x-t)^{n+1}(-1)}{n+1} \right]_{t=x_0}^{t=x}$$

$$= \frac{M^{n+1}}{n!}\left[\frac{(-1)(x-x)^{n+1}}{n+1} - \frac{(-1)(x-x_0)^{n+1}}{n+1}\right] =$$

$$= \frac{M^{n+1}}{n!}\,\frac{(x-x_0)^{n+1}}{n+1} = \frac{[M(x-x_0)]^{n+1}}{(n+1)!} \equiv \gamma_n$$

It follows that

$$\forall n \in \mathbb{N}^*: |E_n(x)| \leq \frac{[M(x-x_0)]^{n+1}}{(n+1)!} = \gamma_n$$

and we note that

$$\forall n \in \mathbb{N}^*: \gamma_n > 0$$

and

$$\forall n \in \mathbb{N}^*: \frac{\gamma_{n+1}}{\gamma_n} = \frac{\frac{[M(x-x_0)]^{n+2}}{(n+2)!}}{\frac{[M(x-x_0)]^{n+1}}{(n+1)!}} = \frac{(n+1)!\,[M(x-x_0)]^{n+2}}{(n+2)!\,[M(x-x_0)]^{n+1}} =$$

$$= \frac{M(x-x_0)}{n+2}$$

We may therefore argue that

$$\lim_{x\to+\infty} \frac{1}{x+2} = \lim_{x\to+\infty} \frac{1}{x} = 0 \Rightarrow \lim_{n\in\mathbb{N}^*} \frac{1}{n+2} = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \frac{M(x-x_0)}{n+2} = 0 \Rightarrow \lim_{n\in\mathbb{N}^*} \frac{\gamma_{n+1}}{\gamma_n} = 0 \Rightarrow \lim_{n\in\mathbb{N}^*} \gamma_n = 0$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} E_n(x) = 0 \Rightarrow \lim_{n\in\mathbb{N}^*} \left[f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right] = 0$$

$$\Rightarrow f(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k. \qquad \square$$

↳ We now apply this theorem to establish the convergence of the Taylor series of trigonometric functions and the exponential function.

1) $\boxed{\forall x\in\mathbb{R}: \sin x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}$

Proof

We define $\forall x\in\mathbb{R}: f(x) = \sin x$

Let $x\in\mathbb{R}$ be given., let $n\in\mathbb{N}$ be given, and note that

$$\begin{cases} f^{(2n)}(x) = (-1)^n \sin x \\ f^{(2n+1)}(x) = (-1)^n \cos x \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} |f^{(2n)}(x)| = |(-1)^n \sin x| = |\sin x| \leq 1 \\ |f^{(2n+1)}(x)| = |(-1)^n \cos x| = |\cos x| \leq 1 \end{cases}$$

It follows that

$(\forall n\in\mathbb{N}: \forall x\in\mathbb{R}: |f^{(n)}(x)| \leq 1 = 1^n) \Rightarrow$

$\Rightarrow$ The Taylor series of $\sin x$ converges on $\mathbb{R}$.

and also:

$$\begin{cases} \forall n\in\mathbb{N}: f^{(2n)}(0) = (-1)^n \sin 0 = 0 \\ \forall n\in\mathbb{N}: f^{(2n+1)}(0) = (-1)^n \cos 0 = (-1)^n \end{cases}$$

and therefore:

$$\forall x\in\mathbb{R}: f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

$$= \sum_{n=0}^{+\infty} \left[\frac{f^{(2n)}(0)}{(2n)!} x^{2n} + \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} \right]$$

$$= \sum_{n=0}^{+\infty} \left[0 x^{2n} + \frac{(-1)^n}{(2n+1)!} x^{2n+1} \right] = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

2) $\boxed{\forall x\in\mathbb{R}:\ \cos x=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots}$

Proof

We define $\forall x\in\mathbb{R}:\quad f(x)=\cos x$

Let $x\in\mathbb{R}$ and $n\in\mathbb{N}$ be given, and note that

$$\begin{cases} f^{(2n)}(x)=(-1)^n\cos x \\ f^{(2n+1)}(x)=(-1)^n[-\sin x]=(-1)^{n+1}\sin x \end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases} |f^{(2n)}(x)|=|(-1)^n\cos x|=|\cos x|\le 1 \\ |f^{(2n+1)}(x)|=|(-1)^{n+1}\sin x|=|\sin x|\le 1 \end{cases}$$

It follows that

$(\forall n\in\mathbb{N}:\forall x\in\mathbb{R}:|f^{(n)}(x)|\le 1=1^n)\Rightarrow$

$\Rightarrow$ The Taylor series of $\cos x$ converges on $\mathbb{R}$.

and also

$$\begin{cases} \forall n\in\mathbb{N}:\ f^{(2n)}(0)=(-1)^n\cos 0=(-1)^n \\ \forall n\in\mathbb{N}:\ f^{(2n+1)}(0)=(-1)^{n+1}\sin 0=(-1)^{n+1}\cdot 0=0 \end{cases}$$

and therefore:

$$\forall x\in\mathbb{R}:\ f(x)=\sum_{n=0}^{+\infty}\frac{f^{(n)}(0)}{n!}x^n=$$

$$=\sum_{n=0}^{+\infty}\left[\frac{f^{(2n)}(0)}{(2n)!}x^{2n}+\frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1}\right]$$

$$=\sum_{n=0}^{+\infty}\left[\frac{(-1)^n}{(2n)!}x^{2n}+0\,x^{2n+1}\right]=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

3) $\boxed{\forall x \in \mathbb{R}: \; e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$

Proof

We define $\forall x \in \mathbb{R}: \; f(x) = e^x$

Let $a \in (0, +\infty)$ be given. Let $x \in (-a, a)$ and $n \in \mathbb{N}$ be given, and note that

$f^{(n)}(x) = e^x \Rightarrow$

$\Rightarrow |f^{(n)}(x)| = |e^x| = e^x < e^a < e^{na} = (e^a)^n$

It follows that

$(\forall n \in \mathbb{N}: \forall x \in (-a, a): |f^{(n)}(x)| < (e^a)^n) \Rightarrow$

$\Rightarrow$ The Taylor series of $f(x) = e^x$ converges on $(-a, a)$ for all $a \in (0, +\infty) \Rightarrow$

$\Rightarrow$ The Taylor series of $f(x) = e^x$ converges on $\mathbb{R}$. (1)

Since

$(\forall n \in \mathbb{N}: f^{(n)}(0) = e^0 = 1) \overset{(1)}{\Rightarrow}$

$\Rightarrow \forall x \in \mathbb{R}: f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n =$

$= \sum_{n=0}^{+\infty} \frac{x^n}{n!}$ □

EXAMPLES

a) Show that $\forall x \in \mathbb{R}: \sin x \sin(3x) = \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n-1}(1-2^{2n})}{(2n)!} x^{2n}$

Solution

$$\forall x \in \mathbb{R}: \sin x \cdot \sin(3x) = (1/2)\left[\cos(x-3x) - \cos(x+3x)\right] =$$

$$= (1/2)\left[\cos(-2x) - \cos(4x)\right] = (1/2)\left[\cos(2x) - \cos(4x)\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{+\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} - \sum_{n=0}^{+\infty} (-1)^n \frac{(4x)^{2n}}{(2n)!}\right] =$$

$$= \frac{1}{2}\sum_{n=0}^{+\infty}\left[(-1)^n \frac{2^{2n} x^{2n}}{(2n)!} - (-1)^n \frac{4^{2n} x^{2n}}{(2n)!}\right] =$$

$$= \frac{1}{2}\sum_{n=0}^{+\infty} (-1)^n \frac{(2^{2n} - 4^{2n})}{(2n)!} x^{2n} =$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n}(1-2^{2n})}{2(2n)!} x^{2n} =$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n-1}(1-2^{2n})}{(2n)!} x^{2n}$$

b) Consider the error function:

$$\forall x \in \mathbb{R}: \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Show that

$$\forall x \in \mathbb{R}: \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)\, n!} x^{2n+1}$$

Solution

$$\forall x \in \mathbb{R}: \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\,dt = \frac{2}{\sqrt{\pi}} \int_0^x \left[\sum_{n=0}^{+\infty} \frac{(-t^2)^n}{n!} \right] dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x \left[\sum_{n=0}^{+\infty} \frac{(-1)^n t^{2n}}{n!} \right] dt =$$

$$= \frac{2}{\sqrt{\pi}} \cdot \sum_{n=0}^{+\infty} \left[\int_0^x \frac{(-1)^n t^{2n}}{n!}\,dt \right] =$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{n!} \int_0^x t^{2n}\,dt \right] =$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{n!} \left[\frac{t^{2n+1}}{2n+1} \right]_{t=0}^{t=x} \right] =$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{n!} \left(\frac{x^{2n+1} - 0}{2n+1} \right) \right] =$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)\,n!}\, x^{2n+1}$$

c) Consider the Fresnel integral function

$$S(x) = \int_0^x \sin(t^2)\,dt\,, \quad \forall x \in \mathbb{R}$$

Show that

$$\forall x \in \mathbb{R} = S(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(4n+3)(2n+1)!}\, x^{4n+3}$$

Solution

$$\forall x \in \mathbb{R}: S(x) = \int_0^x \sin(t^2)\,dt = \int_0^x \left[\sum_{n=0}^{+\infty} (-1)^n \frac{(t^2)^{2n+1}}{(2n+1)!} \right] dt$$

$$= \int_0^x \left[\sum_{n=0}^{+\infty} (-1)^n \cdot \frac{t^{4n+2}}{(2n+1)!} \right] dt = \sum_{n=0}^{+\infty} \left[\int_0^x (-1)^n \frac{t^{4n+2}}{(2n+1)!} dt \right]$$

$$= \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{(2n+1)!} \int_0^x t^{4n+2} dt \right] = \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{(2n+1)!} \frac{x^{4n+3}}{4n+3} \right]$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(4n+3)(2n+1)!} x^{4n+3}$$

↳ The previous two examples show that Taylor series expansions can be used to evaluate challenging integrals that cannot be reduced in terms of elementary functions.

(3) → Binomial series

This is a generalization of Newton's binomial expansion. We begin with a preliminary discussion of the binomial expansion. We then provide the generalization to binomial series.

- The Newton binomial expansion reads:

$$\forall a,b \in \mathbb{R} : \forall n \in \mathbb{N}^* : (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{1}$$

$$\text{with } \forall n,k \in \mathbb{N} : \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2}$$

- We can also show that

$$\forall n \in \mathbb{N} : \binom{n}{0} = 1 \wedge \binom{n}{n} = 1 \tag{3}$$

$$\forall n,k \in \mathbb{N}^* : \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{4}$$

- From Eq.(3) and Eq.(4) we immediately conclude that for all $n,k \in \mathbb{N}$, $\binom{n}{k}$ is an integer, and that it can be evaluated recursively via Pascal's triangle:

$n=1$: 1 1	$(a+b)^1 = a+b$
$n=2$: 1 2 1	$(a+b)^2 = a^2+2ab+b^2$
$n=3$: 1 3 3 1	$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$
$n=4$: 1 4 6 4 1	$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$
$n=5$: 1 5 10 10 5 1	$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

- To establish the binomial expansion, we prove Eq.(3) and Eq.(4) and then use them to prove Eq.(1) by induction.

Proof of Eq.(3) and Eq.(4)

Let $n \in \mathbb{N}$ be given. Then

$$\binom{n}{0} = \frac{n!}{0!\,(n-0)!} = \frac{n!}{0!\,n!} = \frac{1}{0!} = \frac{1}{1} = 1$$

$$\binom{n}{n} = \frac{n!}{n!\,(n-n)!} = \frac{1}{(n-n)!} = \frac{1}{0!} = \frac{1}{1} = 1$$

Let $n, k \in \mathbb{N}^*$ be given. Then:

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-(k-1))!} =$$

$$= \frac{n!}{k\,(k-1)!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!} =$$

$$= \frac{n!}{k\,(k-1)!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)(n-k)!} =$$

$$= \frac{n!}{(k-1)!\,(n-k)!}\left[\frac{1}{k} + \frac{1}{n-k+1}\right]$$

$$= \frac{n!}{(k-1)!\,(n-k)!}\left[\frac{(n-k+1)+k}{k\,(n-k+1)}\right] =$$

$$= \frac{n!}{(k-1)!\,(n-k)!}\,\frac{n+1}{k\,(n-k+1)} = \frac{(n+1)!}{k!\,(n-k+1)} =$$

$$= \frac{(n+1)!}{k!\,((n+1)-k)!} = \binom{n+1}{k}$$

Proof of Eq.(1)

Let $a, b \in \mathbb{R}$ be given.

For $n = 1$:

$$(a+b)^1 = a+b = a^1b^0 + a^0b^1 = \binom{1}{0}a^1b^0 + \binom{1}{1}a^0b^1 =$$

$$= \sum_{k=0}^{1}\binom{1}{k}a^{1-k}b^k$$

For $n=m$, we assume that

$$(a+b)^m = \sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k$$

For $n=m+1$, we will show that

$$(a+b)^{m+1} = \sum_{k=0}^{m+1}\binom{m+1}{k}a^{m+1-k}b^k$$

We have:

$$(a+b)^{m+1} = (a+b)(a+b)^m = (a+b)\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k$$

$$= \sum_{k=0}^{m}\binom{m}{k}(a+b)a^{m-k}b^k = \sum_{k=0}^{m}\binom{m}{k}(a^{m-k+1}b^k + a^{m-k}b^{k+1})$$

$$= \sum_{k=0}^{m}\binom{m}{k}a^{m-k+1}b^k + \sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^{k+1}$$

$$= \sum_{k=0}^{m}\binom{m}{k}a^{(m+1)-k}b^k + \sum_{k=1}^{m+1}\binom{m}{k-1}a^{m-(k-1)}b^{(k-1)+1}$$

$$= \sum_{k=0}^{m}\binom{m}{k}a^{(m+1)-k}b^k + \sum_{k=1}^{m+1}\binom{m}{k-1}a^{(m+1)-k}b^k$$

$$= \binom{m}{0}a^{m+1} + \sum_{k=1}^{m}\left[\binom{m}{k}+\binom{m}{k-1}\right]a^{(m+1)-k}b^k + \binom{m}{m}b^{m+1}$$

$$= \binom{m+1}{0}a^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k}a^{(m+1)-k}b^k + \binom{m+1}{m+1}b^{m+1}$$

$$= \sum_{k=0}^{m+1}\binom{m+1}{k}a^{(m+1)-k}b^k.$$

- The binomial expansion can be generalized for real exponents, yielding the <u>binomial series</u>, according to the following theorem:

<u>Thm</u>: $\forall a \in \mathbb{R}: \forall x \in (-1,1): (1+x)^a = \sum_{n=0}^{+\infty} \binom{a}{n} x^n$

with $\forall a \in \mathbb{R}: \binom{a}{0} = 1$, and

$\forall a \in \mathbb{R}: \forall n \in \mathbb{N}^*: \binom{a}{n} = \prod_{k=1}^{n} \left(\frac{a+1-k}{k}\right) =$

$= \frac{a(a-1)(a-2)\cdots(a+1-n)}{n!}$

<u>Proof</u>

Let $a \in \mathbb{R}$ be given and define $\forall x \in (-1,1): f(x) = (1+x)^a$

Let $n \in \mathbb{N}^*$ be given and note that $f(0) = (1+0)^a = 1$ and

$f'(x) = a(1+x)^{a-1}$

$f''(x) = a(a-1)(1+x)^{a-2}$

$\vdots$

$f^{(n)}(x) = a(a-1)\cdots(a+1-n)(1+x)^{a-n}$.

It follows that

$f^{(n)}(0) = a(a-1)\cdots(a+1-n)(1+0)^{a-n} =$

$= a(a-1)\cdots(a+1-n)$, $\forall n \in \mathbb{N}^*$

and therefore the Taylor expansion of $f(x)$ is given by:

$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{+\infty} \frac{a(a-1)\cdots(a+1-n)}{n!} x^n =$

$= \sum_{n=0}^{+\infty} \binom{a}{n} x^n$.

To establish the convergence of this Taylor series on $(-1,1)$ we will use the ratio test and the absolute convergence test. Let $a \in \mathbb{R}$ and $x \in (-1,1)$ be given and define:

$$\forall n \in \mathbb{N}^*: \gamma_n = \binom{a}{n} x^n$$

It follows that

$$\forall n \in \mathbb{N}^*: \left| \frac{\gamma_{n+1}}{\gamma_n} \right| = \left| \frac{\binom{a}{n+1} x^{n+1}}{\binom{a}{n} x^n} \right| = \left| \frac{x^{n+1} \prod_{k=1}^{n+1} \left(\frac{a+1-k}{k} \right)}{x^n \prod_{k=1}^{n} \left(\frac{a+1-k}{k} \right)} \right| =$$

$$= \left| \frac{x \cdot \frac{a+1-(n+1)}{n+1} \prod_{k=1}^{n} \left(\frac{a+1-k}{k} \right)}{\prod_{k=1}^{n} \left(\frac{a+1-k}{k} \right)} \right|$$

$$= \left| \frac{x(a+1-n-1)}{n+1} \right| = |x| \frac{|a-n|}{n+1}$$

Since:

$$\lim_{t \to +\infty} \frac{|a-t|}{t+1} = \lim_{t \to +\infty} \frac{-(a-t)}{t+1} = \lim_{t \to +\infty} \frac{t-a}{t+1} = \lim_{t \to +\infty} \frac{t}{t} = 1$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} \frac{|a-n|}{n+1} = 1 \Rightarrow$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} \left| \frac{\gamma_{n+1}}{\gamma_n} \right| = \lim_{n \in \mathbb{N}^*} \left[|x| \frac{|a-n|}{n+1} \right] = |x|$$

we conclude that

$$x \in (-1,1) \Rightarrow |x| < 1 \Rightarrow \lim_{n \in \mathbb{N}^*} \left| \frac{\gamma_{n+1}}{\gamma_n} \right| < 1 \Rightarrow \sum_{n=0}^{+\infty} |\gamma_n| \text{ converges}$$

$$\Rightarrow \sum_{n=0}^{+\infty} \gamma_n \text{ converges} \Rightarrow \sum_{n=0}^{+\infty} \binom{a}{n} x^n \text{ converges.} \quad \square$$

⟷ Factorial and double factorial

Binomial series should be simplified and written in terms of the factorial $n!$ or the double factorial $n!!$ and we recall the corresponding definitions below:

1) Factorial

$$0! = 1$$
$$n! = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

2) Double factorial

$$0!! = 1$$
$$(2n)!! = \prod_{k=1}^{n} (2k) = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)$$
$$(2n+1)!! = \prod_{k=0}^{n} (2k+1) = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1)$$

Double factorials can be simplified to factorials using the following identities:

$$(2n)!! = 2^n n! \qquad (2n-1)!! = \frac{(2n)!}{2^n n!}$$

EXAMPLE

a) Write the Taylor expansion of

$$\forall x \in (-\infty, 2): \quad f(x) = \frac{1}{\sqrt{2-x}}$$

around $x=0$, and find the radius of convergence

Solution

We have:

$$f(x) = \frac{1}{\sqrt{2-x}} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1-(x/2)}} = \frac{1}{\sqrt{2}} \left[1 + \left(\frac{-x}{2}\right)\right]^{-1/2} =$$

$$= \frac{1}{\sqrt{2}} \sum_{n=0}^{+\infty} \binom{-1/2}{n} \left(\frac{-x}{2}\right)^n = \frac{1}{\sqrt{2}} \sum_{n=0}^{+\infty} \binom{-1/2}{n} \frac{(-1)^n}{2^n} x^n$$

Since:

$$\forall n \in \mathbb{N}^*: \binom{-1/2}{n} = \prod_{k=1}^{n} \frac{-1/2+1-k}{k} = \prod_{k=1}^{n} \frac{1/2-k}{k} = \prod_{k=1}^{n} \frac{1-2k}{2k}$$

$$= \frac{1}{(2n)!!} \prod_{k=1}^{n} (1-2k) = \frac{(-1)^n}{2^n n!} \prod_{k=1}^{n} (2k-1)$$

$$= \frac{(-1)^n}{2^n n!} \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) = \frac{(-1)^n (2n-1)!!}{2^n n!}$$

it follows that

$$f(x) = \frac{1}{\sqrt{2}} \left[1 + \sum_{n=1}^{+\infty} \binom{-1/2}{n} \frac{(-1)^n}{2^n} x^n\right] =$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \sum_{n=1}^{+\infty} \frac{(-1)^n (2n-1)!!}{2^n n!} \frac{(-1)^n}{2^n} x^n$$

$$= \frac{\sqrt{2}}{2} + \sum_{n=1}^{+\infty} \frac{(2n-1)!! \sqrt{2}}{2^{2n+1} n!} x^n$$

To determine the convergence radius, we note that the binomial series requires

$$\left|\frac{-x}{2}\right| < 1 \Leftrightarrow \frac{|-x|}{2} < 1 \Leftrightarrow \frac{|x|}{2} < 1 \Leftrightarrow |x| < 2$$

$$\Leftrightarrow -2 < x < 2 \Leftrightarrow x \in (-2, 2).$$

and therefore:

$$\forall x \in (-2,2): \frac{1}{\sqrt{2-x}} = \frac{\sqrt{2}}{2} + \sum_{n=1}^{+\infty} \frac{(2n-1)!! \cdot \sqrt{2}}{2^{2n+1} n!} x^n.$$

8) Yet another series for π

We will show that $\pi = 4 - \frac{2}{3} - \sum_{n=2}^{+\infty} \frac{(2n-3)!!}{2^{n-2}(2n+1)n!}$

Solution

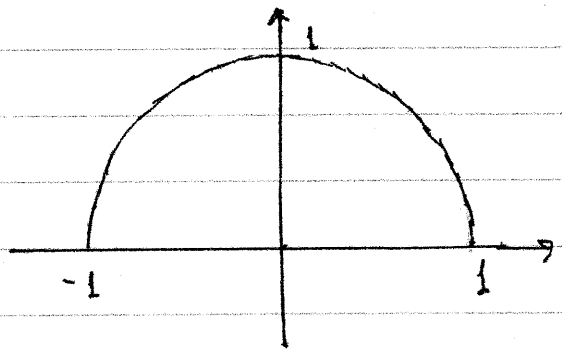
We note that the function

$\forall x \in [-1,1]: f(x) = \sqrt{1-x^2}$

graphs a half-circle with radius 1, because

$x^2 + [f(x)]^2 = x^2 + (\sqrt{1-x})^2 = x^2 + (1-x^2) = 1$, $\forall x \in [-1,1]$

and $\forall x \in [-1,1]: f(x) \geq 0$. Consequently the area over $[-1,1]$ is $\pi/2$, and over $[0,1]$ the area under the graph of $f(x)$ is $\pi/4$. It follows that



$$\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2}\,dx \qquad (1)$$

Note that if we attempt to evaluate this integral using trigonometric substitution, this will only confirm Eq.(1). On the other hand, if we evaluate the integral by term-by-term integration of the Taylor series of $f(x)$, this will give us a series approximation for π.

$$\sqrt{1-x^2} = (1+(-x^2))^{1/2} = \sum_{n=0}^{+\infty} \binom{1/2}{n} (-x^2)^n = \sum_{n=0}^{+\infty} (-1)^n \binom{1/2}{n} x^{2n}, \quad \forall x \in [-1,1]$$

with

$$\forall n\in\mathbb{N}^*:\ \binom{1/2}{n}=\prod_{k=1}^{n}\left(\frac{1/2+1-k}{k}\right)=\prod_{k=1}^{n}\frac{3/2-k}{k}=$$

$$=\prod_{k=1}^{n}\frac{3-2k}{2k}=(-1)^n\prod_{k=1}^{n}\frac{2k-3}{2k}=$$

$$=\frac{(-1)^n}{(2n)!!}\prod_{k=1}^{n}(2k-3)=\frac{(-1)^n}{2^n n!}\left[(-1)\cdot 1\cdot 3\cdot\ldots\cdot(2n-3)\right]$$

$$=\begin{cases}\dfrac{(-1)^{n+1}}{2^n n!}, & \text{if } n=1\\ \dfrac{(-1)^{n+1}(2n-3)!!}{2^n n!}, & \text{if } n>1\end{cases}=$$

$$=\begin{cases}\dfrac{1}{2}, & \text{if } n=1\\ \dfrac{(-1)^{n+1}(2n-3)!!}{2^n n!}, & \text{if } n>1\end{cases}$$

It follows that

$$\frac{\pi}{4}=\int_0^1\sqrt{1-x^2}\,dx=\int_0^1\left[\sum_{n=0}^{+\infty}(-1)^n\binom{1/2}{n}x^{2n}\right]dx=$$

$$=\sum_{n=0}^{+\infty}\left[(-1)^n\binom{1/2}{n}\int_0^1 x^{2n}\,dx\right]=\sum_{n=0}^{+\infty}\left[(-1)^n\binom{1/2}{n}\left[\frac{x^{2n+1}}{2n+1}\right]_0^1\right]=$$

$$=\sum_{n=0}^{+\infty}\left[\frac{(-1)^n}{2n+1}\binom{1/2}{n}\right]=$$

$$=\frac{(-1)^0}{2\cdot 0+1}\binom{1/2}{0}+\frac{(-1)^1}{2\cdot 1+1}\binom{1/2}{1}+\sum_{n=2}^{+\infty}\frac{(-1)^n}{2n+1}\binom{1/2}{n}=$$

$$=\frac{1}{1}\cdot 1+\frac{-1}{3}\,\frac{1}{2}+\sum_{n=2}^{+\infty}\frac{(-1)^n}{2n+1}\,\frac{(-1)^{n+1}(2n-3)!!}{2^n n!}=$$

$$= 1 - \frac{1}{6} + \sum_{n=2}^{+\infty} \frac{(-1)^{2n+1}(2n-3)!!}{(2n+1)2^n n!} =$$

$$= 1 - \frac{1}{6} - \sum_{n=2}^{+\infty} \frac{(2n-3)!!}{(2n+1)2^n n!} \Rightarrow$$

$$\Rightarrow \pi = 4 - \frac{2}{3} - \sum_{n=2}^{+\infty} \frac{4(2n-3)!!}{(2n+1)2^n n!} =$$

$$= 4 - \frac{2}{3} - \sum_{n=2}^{+\infty} \frac{(2n-3)!!}{2^{n-2}(2n+1)n!}$$

↳ Unfortunately this series converges slowly. Using 6 terms:

$$a_6 = 4 - \frac{2}{3} - \frac{1!!}{2^0 \cdot 5 \cdot 2!} - \frac{3!!}{2^1 \cdot 7 \cdot 3!} - \frac{5!!}{2^2 \cdot 9 \cdot 4!} - \frac{7!!}{2^3 \cdot 11 \cdot 5!}$$

$$= 4 - \frac{2}{3} - \frac{1}{10} - \frac{3 \cdot 1}{2 \cdot 7 \cdot 3 \cdot 2 \cdot 1} - \frac{5 \cdot 3}{4 \cdot 9 \cdot 4 \cdot 3 \cdot 2 \cdot 1} -$$

$$- \frac{7 \cdot 5 \cdot 3}{8 \cdot 11 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} =$$

$$= 4 - \frac{2}{3} - \frac{1}{10} - \frac{1}{28} - \frac{5}{288} - \frac{7}{704}$$

$$= \frac{703049}{221760} = 3.1703\ldots \leftarrow \text{first decimal converged}$$

(4) → Product of series

Given power series for two functions $f(x)$ and $g(x)$, we may calculate the power series for $f(x)g(x)$ according to the following theory.

Def: Let a_n, b_n be two sequences with $n \in \mathbb{N}$. We define the Cauchy product $c_n = (a*b)_n$ as:

$$\forall n \in \mathbb{N}: \quad (a*b)_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Thm: (Merten's theorem)

Let a_n, b_n be two sequences with $n \in \mathbb{N}$. Then:

$$\left\{ \begin{array}{l} \sum_{n=0}^{+\infty} |a_n| \text{ converges} \\ \sum_{n=0}^{+\infty} b_n \text{ converges} \end{array} \right. \Rightarrow \left[\sum_{n=0}^{+\infty} a_n \right] \left[\sum_{n=0}^{+\infty} b_n \right] = \sum_{n=0}^{+\infty} (a*b)_n$$

↳ It is important to note that the assumption

$$\left\{ \begin{array}{l} \sum_{n=0}^{+\infty} a_n \text{ converges} \\ \sum_{n=0}^{+\infty} b_n \text{ converges} \end{array} \right.$$

is not sufficiently strong to derive the conclusion of Merten's theorem. For example, using:

$$\forall n \in \mathbb{N}: a_n = b_n = \frac{(-1)^n}{\sqrt{1+n}}$$

we can prove, via the alternating test that $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ converges, but that $\sum_{n=0}^{+\infty} (a*b)_n$ diverges. Details are given in the counterexample below.

↳ Since any convergent power series converges absolutely (and also uniformly), we can always use Merten's theorem to multiply power series. It follows that given an interval $A = (x_0 - r, x_0 + r)$ as well as:

$$\forall x \in A: \left(f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n \wedge g(x) = \sum_{n=0}^{+\infty} b_n (x-x_0)^n\right)$$

using Merten's theorem, it follows that

$$\forall x \in A: f(x)g(x) = \left[\sum_{n=0}^{+\infty} a_n (x-x_0)^n\right]\left[\sum_{n=0}^{+\infty} b_n (x-x_0)^n\right] =$$

$$= \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n}\left[a_k (x-x_0)^k\right]\left[b_{n-k}(x-x_0)^{n-k}\right]\right]$$

$$= \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} a_k b_{n-k} (x-x_0)^n\right]$$

$$= \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right](x-x_0)^n$$

and therefore:

$$\boxed{\left[\sum_{n=0}^{+\infty} a_n (x-x_0)^n\right]\left[\sum_{n=0}^{+\infty} b_n (x-x_0)^n\right] = \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right](x-x_0)^n}$$

EXAMPLE

a) Write the Taylor expansion for $f(x) = e^x \cos x$ around $x=0$ and find the radius of convergence.

Solution

$$f(x) = e^x \cos x = \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right] =$$

$$= \left[\sum_{n=0}^{+\infty} \left(\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}\right)\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right]$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right] + \left[\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right]$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2k}}{(2k)!} (-1)^{n-k} \frac{x^{2n-2k}}{(2n-2k)!}\right] +$$

$$+ \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{x^{2n-2k}}{(2n-2k)!}\right]$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(2k)!\,(2n-2k)!}\right] x^{2n}$$

$$+ \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(2k+1)!\,(2n-2k)!}\right] x^{2n+1}, \quad \forall x \in \mathbb{R}.$$

The reason why the above series converges on $\mathbb{R}$ is because the series expansions of both e^x and $\cos x$ converge on $\mathbb{R}$.

↳ Note that due to the fact that the expansion of $\cos x$ has only even terms, it becomes necessary for $f(x)$ to separate the even from the odd powers of x.

b) Consider the sequence $\forall n \in \mathbb{N}^*: a_n = \frac{(-1)^n}{\sqrt{1+n}}$ and show that $\sum_{n=0}^{+\infty} a_n$ converges & $\sum_{n=0}^{+\infty} (a*a)_n$ diverges.

Solution

- To show that $\sum_{n=0}^{+\infty} a_n$ converges:

Define $\forall n \in \mathbb{N}^*: b_n = \frac{1}{\sqrt{1+n}}$

We note that $\forall n \in \mathbb{N}^*: b_n > 0$ (1)

and

$$\lim_{x \to +\infty} (1+x) = \lim_{x \to +\infty} x = +\infty \Rightarrow \lim_{x \to +\infty} \sqrt{1+x} = +\infty \Rightarrow \lim_{x \to +\infty} \frac{1}{\sqrt{1+x}} = 0$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} b_n = \lim_{n \in \mathbb{N}^*} \frac{1}{\sqrt{1+n}} = 0 \quad (2)$$

We claim that b_n decreasing.

Let $n \in \mathbb{N}^*$ be given. Then:

$$n+1 > n \Rightarrow (n+1)+1 > n+1 > 0 \Rightarrow \sqrt{(n+1)+1} > \sqrt{n+1} > 0$$

$$\Rightarrow \frac{1}{\sqrt{(n+1)+1}} < \frac{1}{\sqrt{n+1}} \Rightarrow b_{n+1} < b_n$$

and it follows that

$(\forall n \in \mathbb{N}^*: b_{n+1} < b_n) \Rightarrow b_n$ decreasing (3)

From Eq.(1), Eq.(2), Eq.(3):

$\sum_{n=0}^{+\infty} (-1)^n b_n = \sum_{n=0}^{+\infty} \frac{(-1)^n}{\sqrt{1+n}}$ converges.

- To show that $\sum_{n=0}^{+\infty} (a*a)_n$ diverges:

$$\forall n\in\mathbb{N}: |(a*a)_n| = \left|\sum_{k=0}^{n} a_k a_{n-k}\right| = \left|\sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}}\right|$$

$$= \left|\sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{k+1}\sqrt{n-k+1}}\right| = \left|(-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}}\right|$$

$$= \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}} \geq \sum_{k=0}^{n} \frac{1}{\sqrt{n+1}\sqrt{n+1}} =$$

$$= \sum_{k=0}^{n} \frac{1}{n+1} = \frac{n+1}{n+1} = 1$$

$$\Rightarrow (\forall n\in\mathbb{N}: |(a*a)_n| \geq 1) \Rightarrow \lim_{n\in\mathbb{N}} (a*a)_n \neq 0 \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{+\infty} (a*a)_n \text{ diverges.}$$

↳ This counterexample demonstrates that absolute convergence is needed for at least one of the two convergent series in order to guarantee the convergence of the series defined via the Cauchy product.

c) Write the Taylor expansion of

$$\forall x \in \mathbb{R} - \{-1/2\} : f(x) = \frac{e^x}{2x+1}$$

around $x=0$, and find the radius of convergence.

Solution

We have:

$$f(x) = \frac{e^x}{2x+1} = e^x \cdot \frac{1}{1-(-2x)} = \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-2x)^n\right] =$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-2)^n x^n\right] =$$

$$= \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} \frac{1}{k!} \cdot (-2)^{n-k}\right] x^n =$$

$$= \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} \frac{(-2)^{n-k}}{k!}\right] x^n$$

For convergence, we require

$$|-2x| < 1 \Leftrightarrow |2x| < 1 \Leftrightarrow 2|x| < 1 \Leftrightarrow |x| < 1/2 \Leftrightarrow$$

$$\Leftrightarrow x \in (-1/2, +1/2).$$

d) Write the Taylor expansion of

$$\forall x \in (-1,+\infty): \quad f(x) = \frac{\ln(x+1)}{x^2+2x+1}$$

around $x=0$ and find the radius of convergence.

Solution

Since:

$$f(x) = \frac{\ln(x+1)}{x^2+2x+1} = \frac{\ln(x+1)}{(x+1)^2} = [\ln(x+1)](x+1)^{-2} =$$

$$= \left[\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}x^n}{n}\right]\left[\sum_{n=0}^{+\infty}\binom{-2}{n}x^n\right] =$$

$$= x\left[\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}x^{n-1}}{n}\right]\left[\sum_{n=0}^{+\infty}\binom{-2}{n}x^n\right]$$

$$= x\left[\sum_{n=0}^{+\infty} \frac{(-1)^{n+2}x^n}{n+1}\right]\left[\sum_{n=0}^{+\infty}\binom{-2}{n}x^n\right]$$

$$= x\left[\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\binom{-2}{k}\frac{(-1)^{n-k}}{n-k+1}\right)x^n\right]$$

$$= \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n}\binom{-2}{k}\frac{(-1)^{n-k}}{n-k+1}\right]x^{n+1}$$

with

$$\binom{-2}{k} = \prod_{a=1}^{k}\frac{-2+1-a}{a} = \prod_{a=1}^{k}\frac{-1-a}{a} = (-1)^k\prod_{a=1}^{k}\frac{a+1}{a}$$

$$= (-1)^k\frac{2\cdot 3\cdot \ldots \cdot (k+1)}{1\cdot 2\cdot \ldots \cdot k} = \frac{(-1)^k(k+1)!}{k!} =$$

$$= \frac{(-1)^k(k+1)k!}{k!} = (-1)^k(k+1)$$

it follows that

$$f(x) = \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} (-1)^k (k+1) \frac{(-1)^{n-k}}{n-k+1} \right] x^{n+1} =$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^n (k+1)}{n-k+1} \right] x^{n+1} =$$

$$= \sum_{n=0}^{+\infty} (-1)^n \left[\sum_{k=0}^{n} \frac{k+1}{n-k+1} \right] x^{n+1}$$

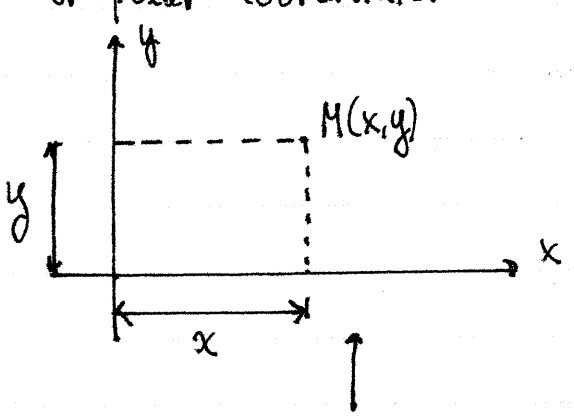
Convergence requires $x \in (-1, 1)$ for the expansion of $\ln(x+1)$ and $(x+1)^{-2}$.

CAL 2.7: Parametric Curves

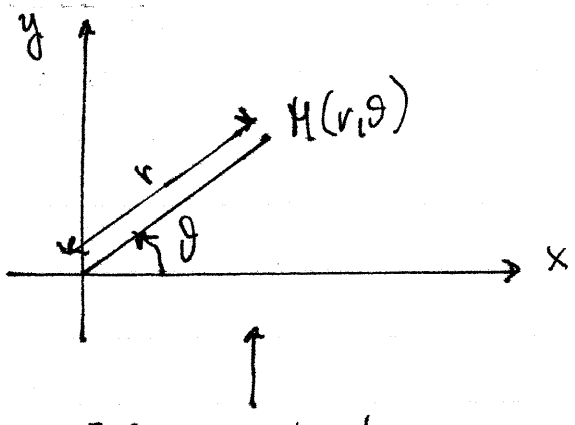
PARAMETRIC CURVES

Definition of parametric curves.

- Parametric curves can be defined either in cartesian or polar coordinates



Cartesian coordinates



Polar coordinates

- The relationship between cartesian coordinates (x,y) and polar coordinates (r,ϑ) is given by:

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$$

and conversely, by:

$r = \sqrt{x^2+y^2}$	$\vartheta = \begin{cases} \text{Arctan}(y/x) & \text{, if } x>0 \\ \pi - \text{Arctan}(y/x) & \text{, if } x<0 \\ \pi/2 & \text{, if } x=0 \wedge y>0 \\ 3\pi/2 & \text{, if } x=0 \wedge y<0 \end{cases}$

(1) → Cartesian curves

The parametric curve (c) given by:

$$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$$

is defined as the set of points

$$(c) = \{(f(t), g(t)) \mid t \in [a,b]\}$$

and the corresponding belonging condition is

$$(x,y) \in (c) \iff \exists t \in [a,b] : (x = f(t) \wedge y = g(t))$$

(2) → Polar curves

The parametric curve (c) given by

$$(c): r = f(\vartheta), \quad \vartheta \in [a,b]$$

is defined as the cartesian parametric curve

$$(c): \begin{cases} x = f(\vartheta)\cos\vartheta \\ y = f(\vartheta)\sin\vartheta \end{cases}, \quad \vartheta \in [a,b]$$

The corresponding set of points is given by

$$(c) = \{(f(\vartheta)\cos\vartheta, f(\vartheta)\sin\vartheta) \mid \vartheta \in [a,b]\}$$

and its belonging condition is given by

$$(x,y) \in (c) \iff \exists \vartheta \in [a,b] : (x = f(\vartheta)\cos\vartheta \wedge y = f(\vartheta)\sin\vartheta)$$

EXAMPLES

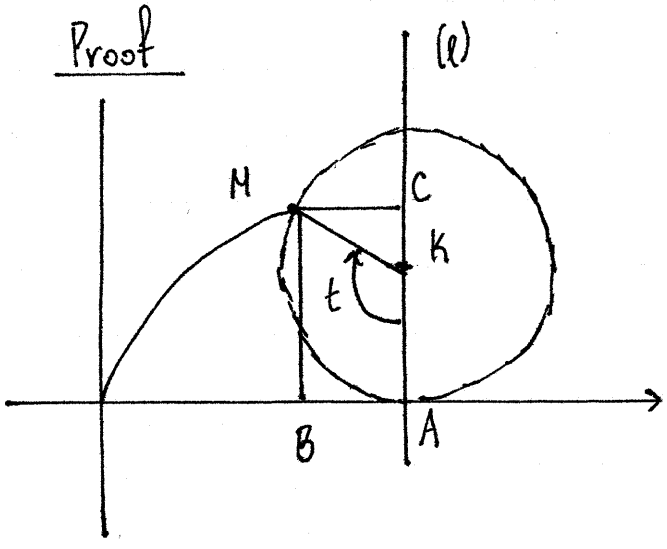
a) The circle with center (x_0, y_0) and radius r is given by:

$$(c): \begin{cases} x = r\cos\vartheta + x_0 \\ y = r\sin\vartheta + y_0 \end{cases}, \quad \vartheta \in [0, 2\pi]$$

b) The cycloid is the curve traced by a fixed point on a circle if the circle "rolls" forward on the x-axis. If the circle has radius a, then the cycloid is given by

$$(c): \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, \quad t \in \mathbb{R}$$

Proof



Let O be the origin, K be the center of the rolling circle, and M a fixed point on the circle. Let A be the point of contact between the circle and the x-axis. Let B the projection of M onto the x-axis.

Let C be the projection of M onto the line (ℓ) defined by the points K and A. We note that

$OA = \text{arc } \overset{\frown}{AM} = at$

$MC = MK \sin(M\hat{K}C) = a \sin(\pi - t) = a \sin t$

$KC = MK \cos(M\hat{K}C) = a \cos(\pi - t) = -a \cos t$

It follows that if $M(x,y)$ (i.e. the point M has coordinates (x,y)) then:

$x = OB = OA - AB = OA - MC = at - a \sin t = a(t - \sin t)$

$y = BM = AC = AK + KC = a - a \cos t = a(1 - \cos t)$. □

Calculus on parametric evaluations

- Consider a parametric curve (c) given by

$$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$$

Then, we define the cartesian derivative dy/dx as a function of the parameter t with $t \in A$ and $A \subseteq [a,b]$ such that

$$\forall t \in A: \quad \left.\frac{dy}{dx}\right|_t = \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{f(t+\Delta t) - f(t)}$$

- Our main result is the following theorem:

Consider a parametric curve (c) given by

$$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$$

and assume that

$$\begin{cases} f, g \text{ differentiable on } [a,b] \\ f', g' \text{ continuous on } [a,b] \\ \forall t \in (a,b): f'(t) \neq 0 \end{cases}$$

Then, it follows that:

a) There is a function $F: f([a,b]) \longrightarrow \mathbb{R}$ such that $\forall (x,y) \in (c): F(x) = y$

b) $\forall t \in [a,b]: \left.\frac{dy}{dx}\right|_t = F'(f(t)) = \frac{g'(t)}{f'(t)}$

c) $\forall t_1, t_2 \in [a,b]: \int_{f(t_1)}^{f(t_2)} F(x)\,dx = \int_{t_1}^{t_2} g(t) f'(t)\,dt$

Proof

We need the following preliminary claim:

$(\forall t \in (a,b): f'(t) > 0) \vee (\forall t \in (a,b): f'(t) < 0)$

To show the claim, we assume the opposite statement

$(\exists t \in (a,b): f'(t) < 0) \wedge (\exists t \in (a,b): f'(t) > 0)$

Fix $t_1, t_2 \in (a,b)$ such that $f'(t_1) > 0$ and $f'(t_2) < 0$ and assume, with no loss of generality, that $t_1 < t_2$. Then

$$\left.\begin{array}{l} f'(t_1) f'(t_2) < 0 \\ f' \text{ continuous on } [t_1, t_2] \end{array}\right\} \Rightarrow \exists t_0 \in [t_1, t_2]: f'(t_0) = 0$$

which is a contradiction since $\forall t \in (a,b): f'(t) \neq 0$.

This proves a claim.

- Proof of (a): Assume, with no loss of generality that,

$(\forall t \in (a,b): f'(t) > 0) \Rightarrow f$ strictly increasing on $[a,b]$

$\Rightarrow f$ has an inverse function f^{-1}.

Using the, now established, f^{-1}, we define $F: f([a,b]) \to \mathbb{R}$:

$\forall x \in f([a,b]): F(x) = (g \circ f^{-1})(x)$

We will now show that $\forall (x,y) \in (c): F(x) = y$.

Let $(x,y) \in (c)$ be given. Then $\exists t \in [a,b]: (x = f(t) \wedge y = g(t))$.

Fix a $t \in [a,b]$ such that $x = f(t)$ and $y = g(t)$. Then $t = f^{-1}(x)$ and it follows that

$$y = g(t) = g(f^{-1}(x)) = (g \circ f^{-1})(x) = F(x).$$

We have thus shown that $\forall (x,y) \in (c): y = F(x)$.

- Proof of (b)

Let $t \in [a,b]$ be given. Then

$$g'(t) = (g \circ id([a,b])'(t) = (g \circ (f^{-1} \circ f))'(t) = ((g \circ f^{-1}) \circ f)'(t) =$$

$$= (F \circ f)'(t) = F'(f(t))\, f'(t) \Rightarrow F'(f(t)) = \frac{g'(t)}{f'(t)}$$

and

$$\left.\frac{dy}{dx}\right|_t = \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{f(t+\Delta t) - f(t)} = \lim_{\Delta t \to 0} \frac{\left(\frac{g(t+\Delta t) - g(t)}{\Delta t}\right)}{\left(\frac{f(t+\Delta t) - f(t)}{\Delta t}\right)} =$$

$$= \frac{\lim\limits_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}}{\lim\limits_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}} = \frac{g'(t)}{f'(t)}$$

It follows that

$$\forall t \in [a,b]: \left.\frac{dy}{dx}\right|_t = F'(f(t)) = \frac{g'(t)}{f'(t)}$$

• Proof of (c)

Let $t_1, t_2 \in [a,b]$ be given. Then:

$$I = \int_{f(t_1)}^{f(t_2)} F(x)\,dx = \int_{f(t_1)}^{f(t_2)} (g \circ f^{-1})(x)\,dx = \int_{f(t_1)}^{f(t_2)} g(f^{-1}(x))\,dx$$

Let $t = f^{-1}(x)$. Then: $x = f(t) \Rightarrow dx = f'(t)\,dt$.

For $x = f(t_1) \Rightarrow t = f^{-1}(f(t_1)) = t_1$

For $x = f(t_2) \Rightarrow t = f^{-1}(f(t_2)) = t_2$

It follows that

$$I = \int_{t_1}^{t_2} g(t) \cdot f'(t)\,dt \qquad \square$$

and therefore

$$\forall t_1, t_2 \in [a,b]: \int_{f(t_1)}^{f(t_2)} F(x)\,dx = \int_{t_1}^{t_2} g(t) f'(t)\,dt$$

→ Methodology: Derivatives dy/dx and d^2y/dx^2

The previous result for calculating dy/dx can be rewritten as:

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

Applying this result onto itself gives the second derivative d^2y/dx^2 which is given by:

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}}$$

EXAMPLES

(a) Consider the parametric curve

$$(c): \begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}, \quad t \in \mathbb{R}$$

a) Find all points where (c) has a horizontal tangent.

b) Find the concavity of (c) in terms of $t \in \mathbb{R}$.

Solution

a) $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(d/dt)(t^3-3t)}{(d/dt)t^2} = \frac{3t^2-3}{2t} =$$

$$= \frac{3(t^2-1)}{2t} = \frac{3(t-1)(t+1)}{2t}$$

horizontal

We have a tangent at $t \Leftrightarrow dy/dx = 0 \Leftrightarrow$

$\Leftrightarrow \frac{3(t-1)(t+1)}{2t} = 0 \Leftrightarrow 3(t-1)(t+1) = 0 \Leftrightarrow$

$\Leftrightarrow t-1=0 \vee t+1=0 \Leftrightarrow t=1 \vee t=-1$

For $t=1$: $x=t^2=1^2 \wedge y=t^3-3t=1^3-3\cdot 1=1-3=-2$

thus $(x,y)=(1,-2)$

For $t=-1$: $x=t^2=(-1)^2=1 \wedge y=t^3-3t=(-1)^3-3(-1)=-1+3=2$

thus $(x,y)=(1,2)$

B)

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{(d/dt)t^2} =$$

$$= \frac{1}{2t} \cdot \frac{(3t^2-3)'(2t)-(3t^2-3)(2t)'}{4t^2} =$$

$$= \frac{(6t)(2t)-(3t^2-3)2}{8t^3} = \frac{12t^2-6t^2+6}{8t^3} =$$

$$= \frac{6t^2+6}{8t^3} = \frac{3t^2+3}{4t^3} = \frac{3(t^2+1)}{4t^3}$$

Sign table:

t		0	
$3(t^2+1)$	$+$		$+$
$4t^3$	$-$	0	$+$
d^2y/dx^2	$-$	$\nexists$	$+$
$y=F(x)$	$\cap$		$\cup$

and therefore (c) is concave down for $t \in (-\infty, 0)$ and (c) is concave up for $t \in (0, +\infty)$.

B) Consider the polar curve $(c): r = 1+\sin\vartheta$. Evaluate the derivative dy/dx and find all points where the tangent line to (c) is horizontal.

Solution

We note that

$$(c): \begin{cases} x = (1+\sin\vartheta)\cos\vartheta \\ y = (1+\sin\vartheta)\sin\vartheta \end{cases}, \quad \vartheta \in [0, 2\pi).$$

Then:

$$dx/d\vartheta = [(1+\sin\vartheta)\cos\vartheta]' = (1+\sin\vartheta)'\cos\vartheta + (1+\sin\vartheta)(\cos\vartheta)' =$$
$$= \cos\vartheta\cos\vartheta + (1+\sin\vartheta)(-\sin\vartheta) = \cos^2\vartheta - \sin\vartheta - \sin^2\vartheta$$
$$= 1 - \sin^2\vartheta - \sin\vartheta - \sin^2\vartheta = (1-\sin\vartheta)(1+\sin\vartheta) - \sin\vartheta(1+\sin\vartheta) =$$
$$= (1-\sin\vartheta-\sin\vartheta)(1+\sin\vartheta) = (1-2\sin\vartheta)(1+\sin\vartheta)$$

and

$$dy/d\vartheta = [(1+\sin\vartheta)\sin\vartheta]' = (1+\sin\vartheta)'\sin\vartheta + (1+\sin\vartheta)(\sin\vartheta)' =$$
$$= \cos\vartheta\sin\vartheta + (1+\sin\vartheta)\cos\vartheta = \cos\vartheta[\sin\vartheta + (1+\sin\vartheta)]$$
$$= \cos\vartheta(2\sin\vartheta + 1).$$

and therefore

$$\frac{dy}{dx} = \frac{dy/d\vartheta}{dx/d\vartheta} = \frac{\cos\vartheta(2\sin\vartheta+1)}{(1-2\sin\vartheta)(1+\sin\vartheta)}$$

The tangent line to (c) at ϑ is horizontal $\iff dy/dx = 0$

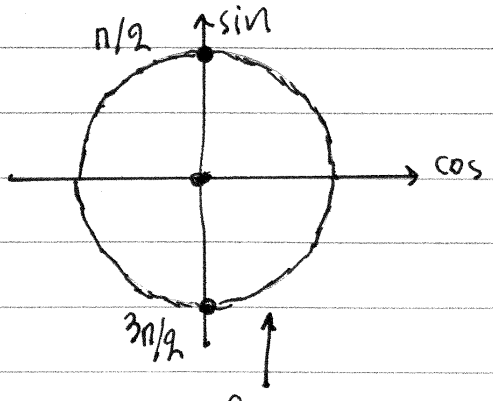
$\iff \dfrac{\cos\vartheta(2\sin\vartheta+1)}{(1-2\sin\vartheta)(1+\sin\vartheta)} = 0$ ← Require $(1-2\sin\vartheta)(1+\sin\vartheta) \neq 0$

$\iff \cos\vartheta(2\sin\vartheta+1) = 0 \iff \cos\vartheta = 0 \ \vee \ 2\sin\vartheta + 1 = 0$

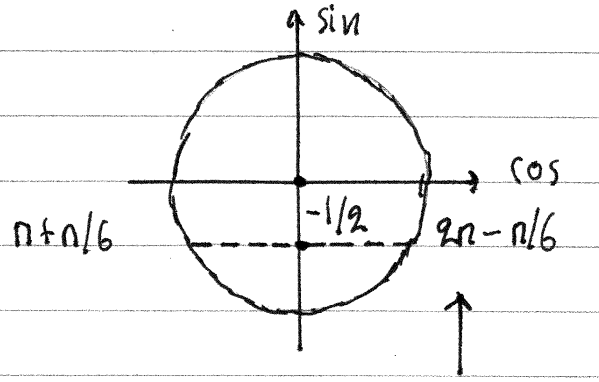
$\iff \cos\vartheta = 0 \ \vee \ \sin\vartheta = \dfrac{-1}{2} = \sin\left(\dfrac{-\pi}{6}\right) \iff$

$\Leftrightarrow (\vartheta = \pi/2 \vee \vartheta = 3\pi/2) \vee (\vartheta = 2\pi - \pi/6 \vee \vartheta = \pi + \pi/6)$

$\Leftrightarrow \vartheta = \pi/2 \vee \vartheta = 3\pi/2 \vee \vartheta = 11\pi/6 \vee \vartheta = 7\pi/6$



$\cos\vartheta = 0$



$\sin\vartheta = -1/2 = \sin(-\pi/6)$

From these solutions, we need to reject all solutions that violate the requirement

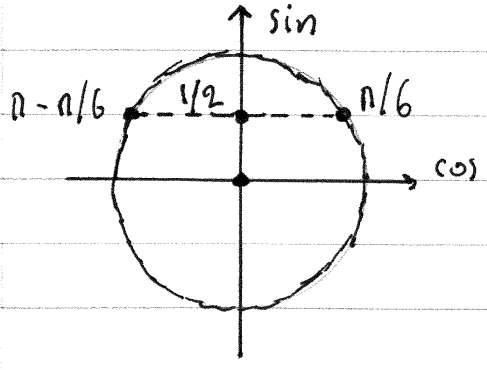
$(1 - 2\sin\vartheta)(1 + \sin\vartheta) \neq 0$

therefore, we solve the equation:

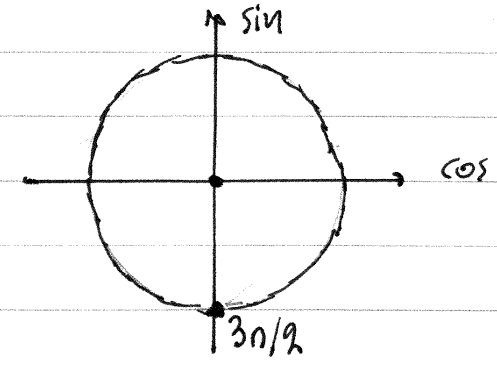
$(1 - 2\sin\vartheta)(1 + \sin\vartheta) = 0 \Leftrightarrow 1 - 2\sin\vartheta = 0 \vee 1 + \sin\vartheta = 0$

$\Leftrightarrow \sin\vartheta = 1/2 = \sin(\pi/6) \vee \sin\vartheta = -1 = \sin(3\pi/2)$

$\Leftrightarrow (\vartheta = \pi/6 \vee \vartheta = 5\pi/6) \vee \vartheta = 3\pi/2$



$\sin\vartheta = 1/2 = \sin(\pi/6)$



$\sin\vartheta = -1 = \sin(3\pi/2)$

It follows that the solution $\vartheta = 3\pi/2$ is rejected and all other solutions are accepted. We conclude that:

$dy/dx = 0 \Leftrightarrow \vartheta \in \{\pi/2, 11\pi/6, 7\pi/6\}$.

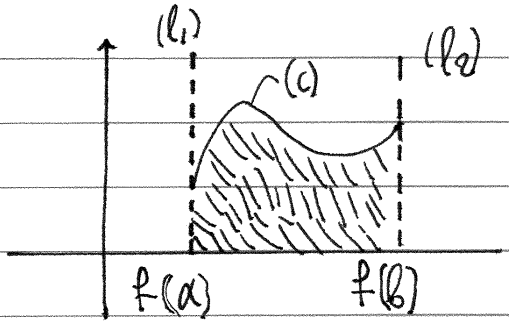
Methodology: Area under the curve (c).

From the previous theorem, the area between the curve (c) defined as

$$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$$

the x-axis, and the lines $(l_1): x = f(a)$ and $(l_2): x = f(b)$ is given by

$$\boxed{A = \int_a^b g(t) f'(t)\, dt}$$



EXAMPLE

Find the area under the curve (c) (the cycloid) defined as

$$(c): \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, \quad t \in [0, 2\pi]$$

from $t = 0$ to $t = 2\pi$.

Solution

Define $\forall t \in [0, 2\pi]: \begin{cases} f(t) = a(t - \sin t) \\ g(t) = a(1 - \cos t) \end{cases}$

Then the area under (c) from $t = 0$ to $t = 2\pi$ is:

$$A = \int_0^{2\pi} g(t) f'(t)\, dt = \int_0^{2\pi} a(1-\cos t)[a(t-\sin t)]'\, dt =$$

$$= \int_0^{2\pi} a(1-\cos t)\, a(1-\cos t)\, dt = a^2 \int_0^{2\pi} (1-\cos t)^2\, dt =$$

$$= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt =$$

$$= a^2 \int_0^{2\pi} (1 - 2\cos t)\, dt + a^2 \int_0^{2\pi} \frac{1 + \cos(2t)}{2}\, dt$$

$$= a^2 \left[t - 2\sin t \right]_0^{2\pi} + \frac{a^2}{2} \left[t + \frac{\sin(2t)}{2} \right]_0^{2\pi} =$$

$$= a^2 \left[(2\pi - 2\sin(2\pi)) - (0 - 2\sin 0) \right] + \frac{a^2}{2} \left[\left(2\pi + \frac{\sin(4\pi)}{2} \right) - \left(0 + \frac{\sin 0}{2} \right) \right]$$

$$= a^2 \left[(2\pi - 0) - (0 - 0) \right] + \frac{a^2}{2} \left[(2\pi + 0) - (0 + 0) \right] =$$

$$= 2\pi a^2 + \pi a^2 = 3\pi a^2 .$$

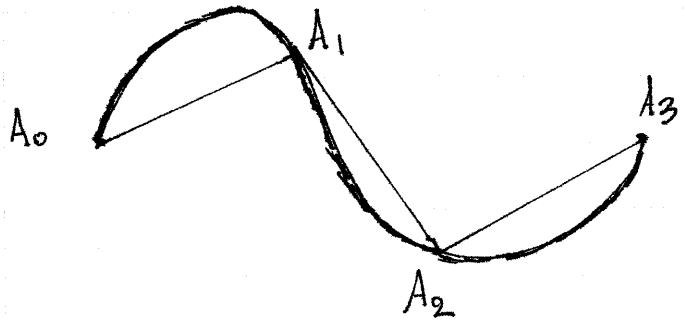
Arclength of parametric curve

- Consider a curve (c) given by

$$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$$

We subdivide the interval $[a,b]$ into n equal subintervals $[t_{k-1}, t_k]$ with $k \in [n]$ such that $a = t_0 < t_1 < \cdots < t_n = b$ with $\forall k \in \{0\} \cup [n] : t_k = a + (b-a)(k/n)$.
We can the approximate the length of the curve (c) with the length of a polygonal line passing through the points $A_k(f(t_k), g(t_k))$ with $k \in \{0\} \cup [n]$, whose length is given by:

$$L_n(c) = \sum_{k=1}^{n} \sqrt{(f(t_k) - f(t_{k-1}))^2 + (g(t_k) - g(t_{k-1}))^2}$$



Def: We say that the curve (c) is **rectifiable** if and only if the sequence $L_n(c)$ converges. If it does, then the **arclength** $\ell(c)$ of (c) is defined as:

$$\ell(c) = \lim_{n \in \mathbb{N}^*} L_n(c)$$

Thm: Let (c) be the curve defined as

$$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$$

and assume that:

$\begin{cases} f, g \text{ differentiable on } [a,b] \\ f', g' \text{ continuous on } [a,b] \end{cases}$

Then (c) is rectifiable with arclength given by

$$\ell(c) = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Proof

The polygonal approximation to the arclength of the curve (c) is given by:

$$\forall n \in \mathbb{N}^*: L_n(c) = \sum_{k=1}^{n} \sqrt{[f(t_n(k)) - f(t_n(k-1))]^2 + [g(t_n(k)) - g(t_n(k-1))]^2}$$

with

$$\forall n \in \mathbb{N}^*: \forall k \in \{0\} \cup [n]: t_n(k) = a + (b-a)(k/n)$$

Let $n \in \mathbb{N}^*$ be given. Using the mean-value theorem, we have:

$$f, g \text{ differentiable on } [a,b] \Rightarrow \forall k \in [n]: \begin{cases} f \text{ differentiable on } [t_n(k-1), t_n(k)] \\ g \text{ differentiable on } [t_n(k-1), t_n(k)] \end{cases}$$

$$\Rightarrow \forall k \in [n]: \exists \tau_n(k) \in (t_n(k-1), t_n(k)): \begin{cases} f(t_n(k)) - f(t_n(k-1)) = f'(\tau_n(k))(t_n(k) - t_n(k-1)) \\ g(t_n(k)) - g(t_n(k-1)) = g'(\tau_n(k))(t_n(k) - t_n(k-1)) \end{cases}$$

We note that $t_n(k) - t_n(k-1) = (b-a)/n$, and therefore:

$$\ell(C) = \lim_{n\in\mathbb{N}^*} L_n(C) =$$

$$= \lim_{n\in\mathbb{N}^*} \sum_{k=1}^{n} \sqrt{[f(t_n(k)) - f(t_n(k-1))]^2 + [g(t_n(k)) - g(t_n(k-1))]^2}$$

$$= \lim_{n\in\mathbb{N}^*} \sum_{k=1}^{n} \sqrt{[f'(\tau_n(k))(b-a)/n]^2 + [g'(\tau_n(k))(b-a)/n]^2}$$

$$= \lim_{n\in\mathbb{N}^*} \sum_{k=1}^{n} \sqrt{(b-a)^2/n^2}\sqrt{[f'(\tau_n(k))]^2 + [g'(\tau_n(k))]^2}$$

$$= \lim_{n\in\mathbb{N}^*} \left[\frac{b-a}{n} \sum_{k=1}^{n} \sqrt{[f'(\tau_n(k))]^2 + [g'(\tau_n(k))]^2} \right]$$

$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

The very last step follows from the assumption that f', g' continuous on $[a,b]$ which implies that

$$h(t) = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

is integrable on $[a,b]$. □

EXAMPLE

Find the arclength of the cycloid (c) given by

$$(c):\begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases}, \quad t\in[0,2\pi]$$

Solution

Since

$\forall t\in[0,2\pi]$: $dx/dt = a\,(d/dt)(t-\sin t) = a(1-\cos t)$

$\forall t\in[0,2\pi]$: $dy/dt = a\,(d/dt)(1-\cos t) = +a\sin t$

it follows that

$$\forall t\in[0,2\pi]: \left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2 = a^2(1-\cos t)^2 + (+a\sin t)^2 =$$

$$= a^2(1-2\cos t+\cos^2 t)+a^2\sin^2 t$$
$$= a^2 - 2a^2\cos t + a^2(\cos^2 t+\sin^2 t)$$
$$= a^2+2a^2\cos t + a^2 = 2a^2 - 2a^2\cos t =$$
$$= 2a^2(1-\cos t) = 4a^2\sin^2(t/2)$$

and therefore:

$$l(c) = \int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi}\sqrt{4a^2\sin^2(t/2)}\,dt$$

$$= \int_0^{2\pi} 2|a|\,|\sin(t/2)|\,dt = 2|a|\int_0^{2\pi}\sin(t/2)\,dt =$$

$$= 2|a|\left[\frac{-\cos(t/2)}{1/2}\right]_0^{2\pi} = 4\cdot|a|\left[-\cos(t/2)\right]_0^{2\pi} =$$

$$= 4\,|a|\left[\left(-\cos(2\pi/2)\right)-\left(-\cos 0\right)\right] = 4|a|\left[-\cos(\pi)+\cos 0\right]$$
$$= 4\cdot|a|\left[-(-1)+1\right] = 4\cdot|a|\left[1+1\right] = 8\cdot|a|$$

Arclength of a polar curve

Thm: Consider a polar curve $(c): r = f(\vartheta)$, $\vartheta \in [a,b]$ and assume that
$\begin{cases} f \text{ differentiable on } [a,b] \\ f' \text{ continuous on } [a,b] \end{cases}$
Then
$$\ell(c) = \int_a^b \sqrt{[f(\vartheta)]^2 + [f'(\vartheta)]^2}\, d\vartheta$$

Proof

Since, by definition
$$(c): \begin{cases} x = f(\vartheta)\cos\vartheta \\ y = f(\vartheta)\sin\vartheta \end{cases}, \quad \vartheta \in [a,b]$$
it follows that for any $\vartheta \in [a,b]$:

$$dx/d\vartheta = (d/d\vartheta)[f(\vartheta)\cos\vartheta] = f'(\vartheta)\cos\vartheta + f(\vartheta)(\cos\vartheta)' =$$
$$= f'(\vartheta)\cos\vartheta + f(\vartheta)[-\sin\vartheta] = f'(\vartheta)\cos\vartheta - f(\vartheta)\sin\vartheta$$
$$dy/d\vartheta = (d/d\vartheta)[f(\vartheta)\sin\vartheta] = f'(\vartheta)\sin\vartheta + f(\vartheta)(\sin\vartheta)' =$$
$$= f'(\vartheta)\sin\vartheta + f(\vartheta)\cos\vartheta$$

and therefore

$$(dx/d\vartheta)^2 + (dy/d\vartheta)^2 = [f'(\vartheta)\cos\vartheta - f(\vartheta)\sin\vartheta]^2 + [f'(\vartheta)\sin\vartheta + f(\vartheta)\cos\vartheta]^2$$
$$= [f'(\vartheta)]^2\cos^2\vartheta - 2f(\vartheta)f'(\vartheta)\sin\vartheta\cos\vartheta + [f(\vartheta)]^2\sin^2\vartheta$$
$$+ [f'(\vartheta)]^2\sin^2\vartheta + 2f(\vartheta)f'(\vartheta)\sin\vartheta\cos\vartheta + [f(\vartheta)]^2\cos^2\vartheta =$$
$$= [f'(\vartheta)]^2(\cos^2\vartheta + \sin^2\vartheta) + [f(\vartheta)]^2(\cos^2\vartheta + \sin^2\vartheta) =$$
$$= [f'(\vartheta)]^2 + [f(\vartheta)]^2.$$

It follows that the arclength of (c) is given by:

$$\ell(c) = \int_a^b \sqrt{(dx/d\vartheta)^2 + (dy/d\vartheta)^2}\, d\vartheta =$$

$$= \int_a^b \sqrt{[f(\vartheta)]^2 + [f'(\vartheta)]^2}\, d\vartheta \qquad \square$$

EXAMPLE

Consider the curve $(c): r = 1+\sin\vartheta$, $\vartheta \in [0, 2\pi]$. Evaluate the arclength of (c).

Solution

Define $\forall \vartheta \in [0,2\pi]: f(\vartheta) = 1+\sin\vartheta$. Then:

$\forall \vartheta \in [0,2\pi]: f'(\vartheta) = \cos\vartheta$.

and therefore

$$[f(\vartheta)]^2 + [f'(\vartheta)]^2 = (1+\sin\vartheta)^2 + \cos^2\vartheta = 1 + 2\sin\vartheta + \sin^2\vartheta + \cos^2\vartheta =$$
$$= 1 + 2\sin\vartheta + 1 = 2 + 2\sin\vartheta = 2(1+\sin\vartheta) =$$
$$= 2\left[1 + \cos(\pi/2 - \vartheta)\right] = 4\cos^2(\pi/4 - \vartheta/2), \quad \forall \vartheta \in [0, 2\pi].$$

It follows that

$$l(c) = \int_0^{2\pi} \sqrt{[f(\vartheta)]^2 + [f'(\vartheta)]^2}\, d\vartheta = \int_0^{2\pi} \sqrt{4\cos^2(\pi/4 - \vartheta/2)}\, d\vartheta =$$
$$= \int_0^{2\pi} 2\left|\cos(\pi/4 - \vartheta/2)\right| d\vartheta$$

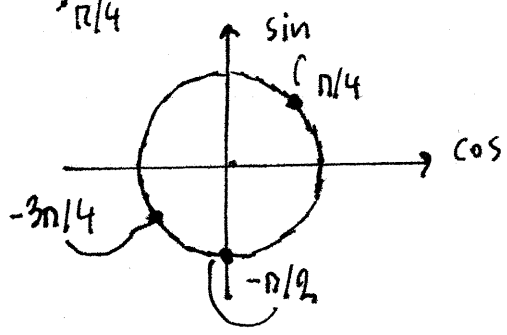
Let $\varphi = \pi/4 - \vartheta/2 \Rightarrow d\varphi = (-1/2)\,d\vartheta \Rightarrow d\vartheta = -2\,d\varphi$

For $\vartheta = 0$: $\varphi = \pi/4 - 0/2 = \pi/4$

For $\vartheta = 2\pi$: $\varphi = \pi/4 - (2\pi)/2 = \pi/4 - \pi = -3\pi/4$

and therefore

$$l(c) = \int_{\pi/4}^{-3\pi/4} 2\left|\cos\varphi\right|(-2)\, d\varphi = 4\int_{-3\pi/4}^{\pi/4} \left|\cos\varphi\right| d\varphi$$



Note that:

$\forall \varphi \in [-3\pi/4, -\pi/2]: \cos\varphi \le 0$

$\forall \varphi \in [-\pi/2, \pi/4]: \cos\varphi \ge 0$

Removing the absolute value gives:

$$l(c) = -4\int_{-3\pi/4}^{-\pi/2} \cos\varphi\, d\varphi + 4\int_{-\pi/2}^{\pi/4} \cos\varphi\, d\varphi = -4\left[\sin\varphi\right]_{-3\pi/4}^{-\pi/2} + 4\left[\sin\varphi\right]_{-\pi/2}^{\pi/4} =$$

$$= -4\left[\sin\left(\frac{-\pi}{2}\right) - \sin\left(\frac{-3\pi}{4}\right)\right] + 4\left[\sin\left(\frac{\pi}{4}\right) - \sin\left(\frac{-\pi}{2}\right)\right] =$$

$$= -4\left[-\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{4}\right)\right] + 4\left[\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right)\right] =$$

$$= -4\left[-1 + \sin(\pi - \pi/4)\right] + 4\left[\frac{\sqrt{2}}{2} + 1\right] =$$

$$= -4\left[-1 + \sin(\pi/4)\right] + 2\sqrt{2} + 4 =$$

$$= -4\left[-1 + \frac{\sqrt{2}}{2}\right] + 2\sqrt{2} + 4 = 4 - 2\sqrt{2} + 2\sqrt{2} + 4 = 8.$$