

LINEAR ALGEBRA

▼ Matrices - Definitions

- An $n \times m$ matrix A is a collection of nm numbers $A_{ab} \in \mathbb{R}$ (with $a \in [n]$ and $b \in [m]$) arranged in n rows and m columns as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{bmatrix} \quad \left| \begin{array}{l} \text{rows } a = 1, 2, \dots, n \\ \text{columns } b = 1, 2, \dots, m \end{array} \right.$$

Remember:

A_{rc} : row, column

A_{vh} : vertical, horizontal

We also write $A = [A_{ab}]$.

A_{ab} = the element of A at row a and column b .

- $M_{nm}(\mathbb{R})$ = the set of all $n \times m$ matrices with elements from $\mathbb{R}$.
- For $n=m$, an $n \times n$ matrix is called a square matrix and we write $M_n(\mathbb{R}) = M_{nn}(\mathbb{R})$.
- Let $A, B \in M_{nm}(\mathbb{R})$ be two matrices, Then $A = B \iff \forall a \in [n] : \forall b \in [m] : A_{ab} = B_{ab}$.
- Zero matrix :
Let $A \in M_{nm}(\mathbb{R})$ be a matrix. Then $A = \mathbf{0} \iff \forall a \in [n] : \forall b \in [m] : A_{ab} = 0$

► Identity Matrix

We say that $I \in M_n(\mathbb{R})$ is an identity matrix if and only if
 $\forall a, b \in [n]: I_{ab} = \begin{cases} 1, & \text{if } a=b \\ 0, & \text{if } a \neq b \end{cases}$

▼ Basic operations with matrices

- Let $A, B, C \in M_{nm}(\mathbb{R})$ be given matrices, and let $\lambda \in \mathbb{R}$.
Then, we define:

$$C = A + B \Leftrightarrow \forall a \in [n]: \forall b \in [m]: C_{ab} = A_{ab} + B_{ab} \quad (\text{addition})$$

$$C = \lambda A \Leftrightarrow \forall a \in [n]: \forall b \in [m]: C_{ab} = \lambda A_{ab}. \quad (\text{scalar multiplication})$$

We also define: $-A = (-1)A$ and $A - B = A + (-1)B$.

- Properties of matrix addition:

$$\forall A, B \in M_{nm}(\mathbb{R}): A + B = B + A$$

$$\forall A, B, C \in M_{nm}(\mathbb{R}): (A + B) + C = A + (B + C)$$

$$\forall A \in M_{nm}(\mathbb{R}): A + \mathbf{0} = \mathbf{0} + A = A$$

$$\forall A \in M_{nm}(\mathbb{R}): \exists B \in M_{nm}(\mathbb{R}): A + B = B + A = \mathbf{0}$$

- Properties of scalar multiplication

$$\forall \lambda \in \mathbb{R}: \forall A, B \in M_{nm}(\mathbb{R}): \lambda(A + B) = \lambda A + \lambda B$$

$$\forall \lambda, \mu \in \mathbb{R}: \forall A \in M_{nm}(\mathbb{R}): \lambda(\mu A) = (\lambda\mu)A$$

$$\forall \lambda, \mu \in \mathbb{R}: \forall A \in M_{nm}(\mathbb{R}): (\lambda + \mu)A = \lambda A + \mu A$$

$$\forall A \in M_{nm}(\mathbb{R}): 1 \cdot A = A$$

$$\forall \lambda \in \mathbb{R}: \lambda \mathbf{0} = \mathbf{0}$$

$$\forall A \in M_{nm}(\mathbb{R}): (-1)A = -A$$

EXAMPLES

a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$.

Calculate $A+B$ and $2A-3B$.

Solution

$$\begin{aligned} A+B &= \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix} = \\ &= \begin{bmatrix} 1+2 & 3+3 & 2+1 \\ 3-1 & 1+0 & 4-2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 3 \\ 2 & 1 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 2A-3B &= 2 \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix} - 3 \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix} = \\ &= \begin{bmatrix} 2 & 6 & 4 \\ 6 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 9 & 3 \\ -3 & 0 & -6 \end{bmatrix} = \\ &= \begin{bmatrix} 2-6 & 6-9 & 4-3 \\ 6-(-3) & 2-0 & 8-(-6) \end{bmatrix} = \begin{bmatrix} -4 & -3 & 1 \\ 9 & 2 & 14 \end{bmatrix} \end{aligned}$$

b) Prove: $\forall A, B, C \in M_{nm}(\mathbb{R}) : (A+B)+C = A+(B+C)$

Solution

Let $A, B, C \in M_{nm}(\mathbb{R})$ be given. Let $\alpha \in [n]$ and $\beta \in [m]$ be given. Then:

$$\begin{aligned} [(A+B)+C]_{\alpha\beta} &= (A+B)_{\alpha\beta} + C_{\alpha\beta} = (A_{\alpha\beta} + B_{\alpha\beta}) + C_{\alpha\beta} = \\ &= A_{\alpha\beta} + (B_{\alpha\beta} + C_{\alpha\beta}) = A_{\alpha\beta} + (B+C)_{\alpha\beta} = \\ &= [A+(B+C)]_{\alpha\beta}. \end{aligned}$$

It follows that

$$\forall a \in [n] : \forall b \in [m] : [(A+B)+C]_{ab} = [A+(B+C)]_{ab}$$

$$\Rightarrow (A+B)+C = A+(B+C)$$

and therefore:

$$\forall A, B, C \in M_{nm}(\mathbb{R}) : (A+B)+C = A+(B+C).$$

c) Prove: $\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : \lambda(\mu A) = (\lambda\mu)A$

Solution

Let $\lambda, \mu \in \mathbb{R}$ and $A \in M_{nm}(\mathbb{R})$ be given. Let $a \in [n]$ and $b \in [m]$ be given. Then

$$[\lambda(\mu A)]_{ab} = \lambda(\mu A)_{ab} = \lambda(\mu A_{ab}) = (\lambda\mu)A_{ab} = [(\lambda\mu)A]_{ab}.$$

It follows that

$$\forall a \in [n] : \forall b \in [m] : [\lambda(\mu A)]_{ab} = [(\lambda\mu)A]_{ab}$$

$$\Rightarrow \lambda(\mu A) = (\lambda\mu)A$$

and therefore

$$\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : \lambda(\mu A) = (\lambda\mu)A.$$

EXERCISES

① Let A, B be the matrices

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 1 & -1 \\ 2 & -4 & 2 \end{bmatrix}$$

a) Evaluate $C = 3A - 2B$

b) Solve with respect to X the equation
 $2A + 3(X - B) = A + B$

② Let A, B be the matrices

$$A = \begin{bmatrix} a & -2a & c \\ 0 & -a & b \\ a+b & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2a & c \\ a & b-a & -b \\ a-b & 0 & -1 \end{bmatrix}$$

Evaluate and simplify $C = A + B$ and
 $D = 2(A - B) - (A + 2B)$

③ Consider the matrix-valued functions

$$A(x) = \begin{bmatrix} 1 & x^2 \\ x & 3x \end{bmatrix}, \quad \forall x \in \mathbb{R}$$

$$B(x) = \begin{bmatrix} x-1 & 2x \\ x^2 & 1 \end{bmatrix}, \quad \forall x \in \mathbb{R}$$

a) Evaluate and simplify the function

$$G(x) = 2A(2x+1) - B(x-2), \quad \forall x \in \mathbb{R}$$

b) Solve with respect to $Y(x)$ the matrix equation

$$3A(x) + 2(Y(x) + A(x)) = A(x+1) - B(x)$$

④ Given the function

$$A(x) = \begin{bmatrix} 1 & 2x & x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$$

Show that

$$A(3x) + 3A(x) = A(0) + 3A(2x), \quad \forall x \in \mathbb{R}.$$

⑤ Prove that

a) $\forall \lambda \in \mathbb{R} : \forall A, B \in M_{n \times n}(\mathbb{R}) : \lambda(A+B) = \lambda A + \lambda B$

b) $\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{n \times n}(\mathbb{R}) : (\lambda + \mu)A = \lambda A + \mu A$

▼ Matrix multiplication

The product AB of two matrices A, B can be defined only when $A \in M_{n \times l}(\mathbb{R})$ and $B \in M_{l \times m}(\mathbb{R})$. That is, the number of columns of A must be equal to the number of rows of B . Then we define the product as follows:

- For $A \in M_{n \times l}(\mathbb{R})$ and $B \in M_{l \times m}(\mathbb{R})$, we define $(AB) \in M_{n \times m}(\mathbb{R})$ such that

$$\forall a \in [n] : \forall b \in [m] : (AB)_{ab} = \sum_{\gamma=1}^l A_{a\gamma} B_{\gamma b}$$

→ To illustrate the definition, we consider the following special cases:

- a) Row matrix $\times$ Column matrix: $A \in M_{1 \times n}(\mathbb{R}) \wedge B \in M_{n \times 1}(\mathbb{R})$.

Then $AB \in M_1(\mathbb{R})$ with

$$AB = [a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} =$$

$$= [a_1 b_1 + a_2 b_2 + \dots + a_n b_n]$$

- b) Product of 2×2 matrices: $A, B \in M_2(\mathbb{R})$.

$$AB = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 c_1 + a_2 d_1 & a_1 c_2 + a_2 d_2 \\ b_1 c_1 + b_2 d_1 & b_1 c_2 + b_2 d_2 \end{bmatrix}$$

↳ From the above examples we see that the element $(AB)_{ab}$ is the product of row a of matrix A and column b of matrix B .

► Properties of Matrix Multiplication

$$\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{kl}(\mathbb{R}) : \forall C \in M_{lm}(\mathbb{R}) : (AB)C = A(BC)$$

$$\forall A \in M_{nk}(\mathbb{R}) : \forall B, C \in M_{km}(\mathbb{R}) : A(B+C) = AB + AC$$

$$\forall B, C \in M_{nk}(\mathbb{R}) : \forall A \in M_{km}(\mathbb{R}) : (B+C)A = BA + CA$$

$$\forall \lambda \in \mathbb{R} : \forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : \lambda(AB) = (\lambda A)B = A(\lambda B)$$

$$\forall A \in M_n(\mathbb{R}) : IA = AI = A \quad (I \in M_n(\mathbb{R}) \text{ is the identity matrix})$$

$$\forall A \in M_n(\mathbb{R}) : A\mathbf{0} = \mathbf{0}A = \mathbf{0}$$

• It is not true for all matrices that $AB = BA$ (see homework for a counterexample). This creates some interesting complications.

► Manipulation Properties

$$\forall A, B, C \in M_{nm}(\mathbb{R}) : A = B \Leftrightarrow A + C = B + C$$

$$\forall A, B \in M_{nk}(\mathbb{R}) : \forall C \in M_{km}(\mathbb{R}) : A = B \Rightarrow AC = BC$$

$$\forall C \in M_{nk}(\mathbb{R}) : \forall A, B \in M_{km}(\mathbb{R}) : A = B \Rightarrow CA = CB$$

$$\forall A, B, C \in M_{nm}(\mathbb{R}) : A + B = C \Leftrightarrow A = C - B$$

• Note that the cancellation property $CA = CB \Rightarrow A = B$ is not true for all matrices

► Matrix powers

Let $A \in M_n(\mathbb{R})$ be a square matrix. We define

$$A^n = \underbrace{A \cdot A \cdot \dots \cdot A}_{n \text{ times}}$$

$$\forall a, b \in \mathbb{N} - \{0\} : \forall A \in M_n(\mathbb{R}) : A^a A^b = A^{a+b}$$

$$\forall a, b \in \mathbb{N} - \{0\} : \forall A \in M_n(\mathbb{R}) : (A^a)^b = A^{ab}$$

EXAMPLES

a) Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. Find all $x, y \in \mathbb{R}$ such that $A^2 = xA - yI$.

Solution

We note that

$$\begin{aligned} A^2 &= AA = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + 1 \cdot 3 & 2 \cdot 1 + 1 \cdot 2 \\ 3 \cdot 2 + 2 \cdot 3 & 3 \cdot 1 + 2 \cdot 2 \end{bmatrix} = \\ &= \begin{bmatrix} 4+3 & 2+2 \\ 6+6 & 3+4 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} xA - yI &= x \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - y \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2x & x \\ 3x & 2x \end{bmatrix} - \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = \\ &= \begin{bmatrix} 2x-y & x \\ 3x & 2x-y \end{bmatrix} \end{aligned}$$

It follows that

$$A^2 = xA - yI \Leftrightarrow \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix} = \begin{bmatrix} 2x-y & x \\ 3x & 2x-y \end{bmatrix} \Leftrightarrow \begin{cases} 2x-y=7 \\ 3x=12 \\ x=4 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2 \cdot 4 - y = 7 \\ x = 4 \end{cases} \Leftrightarrow \begin{cases} 8 - y = 7 \\ x = 4 \end{cases} \Leftrightarrow \begin{cases} y = 8 - 7 = 1 \\ x = 4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x = 4 \wedge y = 1.$$

b) Let $A, B \in M_n(\mathbb{R})$ such that $A^2 = I$ and $B^2 = B$. Show that $(2B - I)^2 = I$ and $(A + I)^2 = 2(A + I)$.

Solution

Assume that $A, B \in M_n(\mathbb{R})$ with $A^2 = I$ and $B^2 = B$. Then

$$\begin{aligned} (2B - I)^2 &= (2B - I)(2B - I) = 2B(2B - I) - I(2B - I) = \\ &= (2B)(2B) - (2B)I - I(2B) + I^2 = \\ &= 4B^2 - 2B - 2B + I = 4B^2 - 4B + I \stackrel{*}{=} \\ &= 4B - 4B + I = 0B + I = 0 + I = I. \end{aligned}$$

and

$$\begin{aligned} (A + I)^2 &= (A + I)(A + I) = A(A + I) + I(A + I) = \\ &= AA + AI + IA + I^2 = A^2 + A + A + I = \\ &= A^2 + 2A + I \stackrel{*}{=} I + 2A + I = 2A + 2I \\ &= 2(A + I). \end{aligned}$$

c) Prove: $\forall B, C \in M_{n \times k}(\mathbb{R}) : \forall A \in M_{k \times m}(\mathbb{R}) : (B + C)A = BA + CA$

Solution

Let $B, C \in M_{n \times k}(\mathbb{R})$ and $A \in M_{k \times m}(\mathbb{R})$ be given. Let $a \in [n]$ and $b \in [m]$ be given. Then

$$\begin{aligned} [(B + C)A]_{ab} &= \sum_{\gamma \in [k]} (B + C)_{a\gamma} A_{\gamma b} = \sum_{\gamma \in [k]} (B_{a\gamma} + C_{a\gamma}) A_{\gamma b} = \\ &= \sum_{\gamma \in [k]} (B_{a\gamma} A_{\gamma b} + C_{a\gamma} A_{\gamma b}) = \\ &= \sum_{\gamma \in [k]} B_{a\gamma} A_{\gamma b} + \sum_{\gamma \in [k]} C_{a\gamma} A_{\gamma b} = \\ &= (BA)_{ab} + (CA)_{ab} = (BA + CA)_{ab} \end{aligned}$$

It follows that

$$\forall a \in [n] : \forall b \in [m] : [(B+C)A]_{ab} = (BA+CA)_{ab} \Rightarrow$$

$$\Rightarrow (B+C)A = BA+CA$$

and therefore

$$\forall B, C \in M_{n \times m}(\mathbb{R}) : \forall A \in M_{m \times m}(\mathbb{R}) : (B+C)A = BA+CA.$$

d) Prove: $\forall A, B, C \in M_{n \times m}(\mathbb{R}) : (A=B \Rightarrow A+C = B+C)$

Solution

Let $A, B, C \in M_{n \times m}(\mathbb{R})$ be given and assume that $A=B$.

Then:

$$A=B \Rightarrow \forall a \in [n] : \forall b \in [m] : A_{ab} = B_{ab} \quad (1)$$

Let $a \in [n]$ and $b \in [m]$ be given. Then:

$$(A+C)_{ab} = A_{ab} + C_{ab} = B_{ab} + C_{ab} = (B+C)_{ab}$$

and it follows that

$$\forall a \in [n] : \forall b \in [m] : (A+C)_{ab} = (B+C)_{ab}.$$

$$\Rightarrow A+C = B+C$$

and therefore we have shown that

$$\forall A, B, C \in M_{n \times m}(\mathbb{R}) : (A=B \Rightarrow A+C = B+C)$$

EXERCISES

⑥ Consider the matrix

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$$

- a) Find the unique $x, y \in \mathbb{R}$ such that $A^2 = xA + yI$
b) Use (a) to find $z, w \in \mathbb{R}$ such that $A^3 = zA + wI$.

⑦ Given the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

evaluate and simplify

- a) $C = AB$ c) $E = 2A^3 - 3A + I$
b) $D = BA - 3B^2$

⑧ Given the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

evaluate and simplify $C = AB - BA$

⑨ Given the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, b, c, d \in \mathbb{R}$

show that: $A^2 - (a+d)A + (ad-bc)I = \mathbf{0}$

⑩ Prove the following properties

a) $\forall A \in M_{n \times k}(\mathbb{R}) : \forall B, C \in M_{k \times m}(\mathbb{R}) : A(B+C) = AB + AC$

b) $\forall A \in M_{n \times k}(\mathbb{R}) : \forall B \in M_{k \times l}(\mathbb{R}) : \forall C \in M_{l \times m}(\mathbb{R}) : (AB)C = A(BC)$

⑪ Rotation matrix

Let $R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Show that

a) $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$, $\forall \theta_1, \theta_2 \in \mathbb{R}$

b) $R(\theta)R(-\theta) = I$, $\forall \theta \in \mathbb{R}$.

⑫ For $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix}$

show that $AB \neq BA$

⑬ For $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ with

$a, b \in \mathbb{R}$, show that $AB = BA$.

⑭ Consider the function

$$\forall x \in \mathbb{R} : M(x) = \begin{bmatrix} 1 & 0 & x \\ -x & 1 & -x^2/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that

$$\forall a, b \in \mathbb{R} : M(a)M(b) = M(a+b).$$

(15) Let $z = a + bi \in \mathbb{C}$ be a complex number, and define

$$M(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Show that

a) $\forall z_1, z_2 \in \mathbb{C} : M(z_1 + z_2) = M(z_1) + M(z_2)$

b) $\forall z_1, z_2 \in \mathbb{C} : M(z_1 z_2) = M(z_1) M(z_2)$

→ This shows that $M(z)$ "imitates" the behaviour of complex number algebra.

(16) Let $A, B \in M_n(\mathbb{R})$. Show that

a) $AB = BA \Rightarrow (A - \lambda I)(B - \lambda I) = (B - \lambda I)(A - \lambda I), \forall \lambda \in \mathbb{R}$

b) $(A+B)^2 = A^2 + 2AB + B^2 \Rightarrow AB = BA$

c) $A^2 = A \Rightarrow (A - I)^2 = I - A$

d) $AB = BA \Rightarrow A^2 B^2 = B^2 A^2$

e) $(B^2 = I \wedge AB = -AB) \Rightarrow AB = BA = 0$

(17) Let $A \in M_n(\mathbb{R})$ such that $A^2 = I$. Show that the matrices

$$B = (1/2)(I + A)$$

$$C = (1/2)(I - A)$$

satisfy $B^2 = B$ and $C^2 = C$.

Matrix Inverses

- Let $A \in M_n(\mathbb{R})$ be a square matrix. We say that B inverse of $A \Leftrightarrow AB = BA = I$.
- We define the set of all matrices $A \in M_n(\mathbb{R})$ that have an inverse as:

$$GL(n, \mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \exists B \in M_n(\mathbb{R}) : AB = BA = I \}$$

We say that

$$A \text{ non-singular} \Leftrightarrow A \in GL(n, \mathbb{R})$$

$$A \text{ singular} \Leftrightarrow A \notin GL(n, \mathbb{R})$$

- Square matrices are not guaranteed to have an inverse. For example, for $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have:

$$\forall x, y, z, w \in \mathbb{R} : \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ z & w \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

However, we will argue that if a matrix does have an inverse, it is unique:

$$\bullet \boxed{\forall A, B, C \in M_n(\mathbb{R}) : \begin{cases} AB = BA = I \\ AC = CA = I \end{cases} \Rightarrow B = C}$$

Proof

Let $A, B, C \in M_n(\mathbb{R})$ be given such that $AB = BA = I$ and $AC = CA = I$. Then

$$\begin{aligned} B &= BI && [\text{Identity matrix}] \\ &= B(AC) && [\text{Hypothesis } AC = I] \\ &= (BA)C && [\text{Associative property}] \end{aligned}$$

$$= I C \quad [\text{Hypothesis } BA = I]$$

$$= C \quad [\text{Identity matrix}]$$

It follows that

$$\forall A, B, C \in M_n(\mathbb{R}) : \begin{cases} AB = BA = I \\ AC = CA = I \end{cases} \Rightarrow B = C \quad \square$$

↳ The unique inverse of A is denoted as A^{-1} , as long as it exists.

► Cancellation property.

$$\boxed{\begin{aligned} \forall A, B \in M_n(\mathbb{R}) : \forall C \in GL(n, \mathbb{R}) : (CA = CB \Leftrightarrow A = B) \\ \forall A, B \in M_n(\mathbb{R}) : \forall C \in GL(n, \mathbb{R}) : (AC = BC \Leftrightarrow A = B) \end{aligned}}$$

Proof

We show only the first statement. Let $A, B \in M_n(\mathbb{R})$ and $C \in GL(n, \mathbb{R})$ be given such that $CA = CB$. Then

$$\begin{aligned} A &= IA = && [\text{identity matrix}] \\ &= (C^{-1}C)A && [C^{-1} \text{ inverse of } C] \\ &= C^{-1}(CA) && [\text{associative property}] \\ &= C^{-1}(CB) && [\text{hypothesis: } CA = CB] \\ &= (C^{-1}C)B && [\text{associative property}] \\ &= IB && [C^{-1} \text{ inverse of } C] \\ &= B && [\text{identity matrix}] \end{aligned}$$

It follows that

$$\forall A, B \in M_n(\mathbb{R}) : \forall C \in GL(n, \mathbb{R}) : (CA = CB \Rightarrow A = B)$$

► Inverse of a 2×2 matrix

• Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$ be a 2×2 square matrix

a) If $D = ad - bc \neq 0$, then A is non-singular with

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

b) If $D = ad - bc = 0$, then A is singular.

► Application to 2×2 linear systems.

Any 2×2 linear system given by

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

can be rewritten in terms of matrix algebra as:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and then solved using the following property:

$$\boxed{\forall A \in GL(n, \mathbb{R}) : \forall x, b \in M_n(\mathbb{R}) : (Ax = b \Leftrightarrow x = A^{-1}b)}$$

Proof

Let $A \in GL(n, \mathbb{R})$ and $x, b \in M_{n1}(\mathbb{R})$ be given. Then:

$$Ax = b \Leftrightarrow A^{-1}(Ax) = A^{-1}b \quad [\text{cancellation property}]$$

$$\Leftrightarrow (A^{-1}A)x = A^{-1}b \quad [\text{associative property}]$$

$$\Leftrightarrow Ix = A^{-1}b \quad [A^{-1} \text{ inverse of } A]$$

$$\Leftrightarrow x = A^{-1}b \quad [\text{identity matrix}]$$

It follows that

$$\forall A \in GL(n, \mathbb{R}) : \forall x, b \in M_{n1}(\mathbb{R}) : (Ax = b \Leftrightarrow x = A^{-1}b) \quad \square$$

Consequently, if $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then:

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases} \Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

From which we may calculate the unique solutions for (x, y) .

↗ Notation: 2x2 determinant

The expression $ad - bc$ is the 2x2 determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and we write:

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

EXAMPLES

a) Use the matrix inverse to solve the system

$$\begin{cases} 2x + 5y = 12 \\ 3x - y = 1 \end{cases}$$

Solution

$$\begin{aligned} \begin{cases} 2x + 5y = 12 \\ 3x - y = 1 \end{cases} &\Leftrightarrow \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \Leftrightarrow \\ \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 1 \end{bmatrix} = \frac{1}{2(-1) - 5 \cdot 3} \begin{bmatrix} -1 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 12 \\ 1 \end{bmatrix} = \\ &= \frac{-1}{17} \begin{bmatrix} (-1) \cdot 12 + (-5) \cdot 1 \\ (-3) \cdot 12 + 2 \cdot 1 \end{bmatrix} = \frac{-1}{17} \begin{bmatrix} -12 - 5 \\ -36 + 2 \end{bmatrix} = \\ &= \frac{-1}{17} \begin{bmatrix} -17 \\ -34 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \text{ Thus solution set } S = \{(1, 2)\} \end{aligned}$$

b) Similarly for the following parametric system:

$$\begin{cases} (a+1)x + (a-1)y = 4a+2 \\ 2ax + (a-1)y = 7a-1 \end{cases}$$

Solution

$$D = \begin{vmatrix} a+1 & a-1 \\ 2a & a-1 \end{vmatrix} = (a+1)(a-1) - 2a(a-1) = (a-1)(a+1-2a) =$$

$$= (a-1)(1-a) = -(a-1)^2.$$

Case 1 : For $a \neq 1 \Rightarrow D \neq 0$, and therefore:

$$\begin{cases} (a+1)x + (a-1)y = 4a+2 \\ 2ax + (a-1)y = 7a-1 \end{cases} \Leftrightarrow \begin{bmatrix} a+1 & a-1 \\ 2a & a-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4a+2 \\ 7a-1 \end{bmatrix} \Leftrightarrow$$

$$\begin{aligned}
 \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} a+1 & a-1 \\ 2a & a-1 \end{bmatrix}^{-1} \begin{bmatrix} 4a+2 \\ 7a-1 \end{bmatrix} = \\
 &= \frac{-1}{(a-1)^2} \begin{bmatrix} a-1 & 1-a \\ -2a & a+1 \end{bmatrix} \begin{bmatrix} 4a+2 \\ 7a-1 \end{bmatrix} = \\
 &= \frac{-1}{(a-1)^2} \begin{bmatrix} (a-1)(4a+2) + (1-a)(7a-1) \\ -2a(4a+2) + (a+1)(7a-1) \end{bmatrix} = \\
 &= \frac{-1}{(a-1)^2} \begin{bmatrix} (a-1)(4a+2-7a+1) \\ -8a^2-4a+7a^2-a+7a-1 \end{bmatrix} = \\
 &= \frac{-1}{(a-1)^2} \begin{bmatrix} (a-1)(-3a+3) \\ -a^2+2a-1 \end{bmatrix} = \\
 &= \frac{-1}{(a-1)^2} \begin{bmatrix} -3(a-1)^2 \\ -(a-1)^2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}
 \end{aligned}$$

$\Leftrightarrow (x, y) = (3, 1)$; thus solution set $S = \{(3, 1)\}$.

Case 2 : For $a=1$; we have:

$$\begin{cases} (1+1)x + (1-1)y = 4 \cdot 1 + 2 \\ 2 \cdot 1x + (1-1)y = 7 \cdot 1 - 1 \end{cases} \Leftrightarrow \begin{cases} 2x = 6 \\ 2x = 6 \end{cases} \Leftrightarrow x = 3$$

which gives as solution set:

$$S = \{(x, y) \in \mathbb{R}^2 \mid x = 3\} = \{(3, y) \mid y \in \mathbb{R}\}.$$

EXERCISES

(18) Use the matrix inverse to solve the following systems:

a) $\begin{cases} x+3y=4 \\ 2x+y=3 \end{cases}$ b) $\begin{cases} 2x-y=3 \\ x+2y=4 \end{cases}$ c) $\begin{cases} 3x+2y=7 \\ x+3y=10 \end{cases}$

d) $\begin{cases} 2ax+(a-3)y=a-1 \\ (a-3)x+2ay=a-a^2 \end{cases}$

e) $\begin{cases} x+(a+1)y=2 \\ (a+2)x+(1-a^2)y=5 \end{cases}$ f) $\begin{cases} ax-y=1-a \\ x-ay=a-a^2 \end{cases}$

→ Distinguish between the values of the parameter $a \in \mathbb{R}$ where the corresponding matrix is non-singular vs. singular.

(19) Find all $a \in \mathbb{R}$ for which the matrix

$$A = \begin{bmatrix} a+3 & 2 \\ 1 & -a \end{bmatrix}$$

i) non-singular.

(20) If $A, B \in M_n(\mathbb{R})$ are non-singular, show that AB is also non-singular with the inverse given by

$$(AB)^{-1} = B^{-1}A^{-1}$$

(21) If $A, B, C \in M_n(\mathbb{R})$ and C is non-singular, then show that

a) $CA = CB \Rightarrow A = B$

b) $AC = BC \Rightarrow A = B$

(22) If $A \in M_n(\mathbb{R})$ with $A^3 = \mathbf{0}$, show that $I - A$ is non-singular with
 $(I - A)^{-1} = I + A + A^2$

(23) If $A \in M_n(\mathbb{R})$ satisfies $A^2 + A + I = \mathbf{0}$, show that A is non-singular and $A^{-1} = A^2$.

(24) If $A, B \in M_n(\mathbb{R})$ with A being non-singular, show that

$$(A - B)A^{-1}(A + B) = (A + B)A^{-1}(A - B)$$

(25) Let $A, B \in M_n(\mathbb{R})$ with $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$. Show that:

$$AB = \mathbf{0} \Rightarrow \begin{cases} A \text{ singular} \\ B \text{ singular} \end{cases}$$

Matrix Transpose

- Let $A \in M_{mn}(\mathbb{R})$ be a matrix. We define the transpose matrix $A^T \in M_{mn}(\mathbb{R})$ as:

$$\forall a \in [m]: \forall b \in [n]: (A^T)_{ab} = A_{ba}$$

- Let $A \in M_n(\mathbb{R})$ be a square matrix. We say that A symmetric $\Leftrightarrow A^T = A \Leftrightarrow \forall a, b \in [n]: A_{ab} = A_{ba}$

Properties

$$\forall A, B \in M_{mn}(\mathbb{R}): (A+B)^T = A^T + B^T$$

$$\forall \lambda \in \mathbb{R}: \forall A \in M_{mn}(\mathbb{R}): (\lambda A)^T = \lambda A^T$$

$$\forall A \in M_{nk}(\mathbb{R}): \forall B \in M_{km}(\mathbb{R}): (AB)^T = B^T A^T$$

$$\forall A \in GL(n, \mathbb{R}): (A^T)^{-1} = (A^{-1})^T$$

$$\forall A \in M_{mn}(\mathbb{R}): (A^T)^T = A$$

EXAMPLES

a) Prove the property

$$\forall A \in M_{nk}(\mathbb{R}): \forall B \in M_{km}(\mathbb{R}): (AB)^T = B^T A^T$$

Solution

Let $A \in M_{nk}(\mathbb{R})$ and $B \in M_{km}(\mathbb{R})$ be given. Let $a \in [n]$ and $b \in [m]$ be given. Then:

$$\begin{aligned} [(AB)^T]_{ab} &= (AB)_{ba} = \sum_{\gamma \in [k]} A_{b\gamma} B_{\gamma a} = \sum_{\gamma \in [k]} A_{\gamma b}^T B_{a\gamma}^T = \\ &= \sum_{\gamma \in [k]} B_{a\gamma}^T A_{\gamma b}^T = (B^T A^T)_{ab} \end{aligned}$$

It follows that

$$\forall a \in [n]: \forall b \in [m]: [(AB)^T]_{ab} = (B^T A^T)_{ab}$$

$$\Rightarrow (AB)^T = B^T A^T$$

and therefore

$$\forall A \in M_{n \times k}(\mathbb{R}): \forall B \in M_{k \times m}(\mathbb{R}): (AB)^T = B^T A^T. \quad \square$$

b) Show that

$$\forall A, B \in M_n(\mathbb{R}): \begin{cases} A, B \text{ symmetric} \\ AB = BA \end{cases} \Rightarrow AB \text{ symmetric.}$$

Solution

Let $A, B \in M_n(\mathbb{R})$ be given such that A, B symmetric and $AB = BA$. Then

$$(AB)^T = B^T A^T$$

[transpose of matrix product]

$$= BA$$

[hypothesis: A, B symmetric]

$$= AB$$

[hypothesis: $AB = BA$]

$\Rightarrow AB$ symmetric.

It follows that

$$\forall A, B \in M_n(\mathbb{R}): \begin{cases} A, B \text{ symmetric} \\ AB = BA \end{cases} \Rightarrow AB \text{ symmetric.}$$

EXERCISES

(26) Give proofs for all properties of the matrix transpose.

(27) Show that if $A \in M_n(\mathbb{R})$ is symmetric and non-singular, then A^{-1} is also symmetric.

(28) Let $A, B \in M_n(\mathbb{R})$. Show that
 A, B, AB symmetric $\Rightarrow AB = BA$.

(29) Given $A, P \in M_n(\mathbb{R})$, show that
 A symmetric $\Rightarrow B = P^T A P$ symmetric

(30) Consider the rotation matrix
$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

a) Show that $R(\theta)$ is non-singular with
 $[R(\theta)]^{-1} = R(-\theta)$

b) Show that
 $[R(\theta)]^T = R(-\theta)$

c) For what angles $\theta \in \mathbb{R}$ is $R(\theta)$ symmetric?

31) Let $A \in M_n(\mathbb{R})$ be a square matrix.

Show that

a) $A + A^T$ symmetric

b) $A^T A$ symmetric

32) Let $z = a + bi \in \mathbb{C}$ with $a, b \in \mathbb{R}$ and i the imaginary unit and define

$$M(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Show that

a) $\forall z \in \mathbb{C} - \{0\} : M(1/z) = M(z)^{-1}$

b) $\forall z_1 \in \mathbb{C} : \forall z_2 \in \mathbb{C} - \{0\} : M(z_1/z_2) = M(z_1)M(z_2)^{-1}$

33) Let $A, B \in M_n(\mathbb{R})$ be two square matrices. We

say that

B skew-symmetric $\Leftrightarrow \forall a, b \in [n] : B_{ab} = -B_{ba}$

$$\Leftrightarrow B^T = -B$$

Show that

$\forall A, B \in M_n(\mathbb{R}) : \begin{cases} A \text{ symmetric} \\ B \text{ skew-symmetric} \end{cases} \Rightarrow A^2 B A^2 \text{ skew-symmetric}$