

LIMITS OF FUNCTIONS

Weierstrass limit definition

Let $f: A \rightarrow \mathbb{R}$ be a function with domain $\text{dom}(f) = A \subseteq \mathbb{R}$.
In order to define $\lim_{x \rightarrow \sigma} f(x) = L$, we begin with the following notation:

Notation

a) The neighborhood $N(\sigma, \delta)$ is defined as

$$N(\sigma, \delta) = \begin{cases} (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) & , \text{ if } \sigma = x_0 \\ (x_0 - \delta, x_0) & , \text{ if } \sigma = x_0^- \\ (x_0, x_0 + \delta) & , \text{ if } \sigma = x_0^+ \\ (1/\delta, +\infty) & , \text{ if } \sigma = +\infty \\ (-\infty, -1/\delta) & , \text{ if } \sigma = -\infty \end{cases}$$

b) The interval $I(L, \varepsilon)$ is defined as

$$I(L, \varepsilon) = \begin{cases} (l - \varepsilon, l + \varepsilon) & , \text{ if } L = l \in \mathbb{R} \\ (1/\varepsilon, +\infty) & , \text{ if } L = +\infty \\ (-\infty, -1/\varepsilon) & , \text{ if } L = -\infty \end{cases}$$

Note that the corresponding belonging conditions are:

$$x \in N(x_0, \delta) \Leftrightarrow 0 < |x - x_0| < \delta$$

$$x \in N(x_0^-, \delta) \Leftrightarrow x_0 - \delta < x < x_0$$

$$x \in N(x_0^+, \delta) \Leftrightarrow x_0 < x < x_0 + \delta$$

$$x \in N(+\infty, \delta) \Leftrightarrow x > 1/\delta$$

$$x \in N(-\infty, \delta) \Leftrightarrow x < -1/\delta$$

$$y \in I(l, \varepsilon) \Leftrightarrow |y - l| < \varepsilon$$

$$y \in I(+\infty, \varepsilon) \Leftrightarrow y > 1/\varepsilon$$

$$y \in I(-\infty, \varepsilon) \Leftrightarrow y < -1/\varepsilon$$

Remarks

a) It is easy to show that

$$\left\{ \begin{array}{l} \forall \delta_1, \delta_2 \in (0, +\infty) : (\delta_1 < \delta_2 \Rightarrow N(\sigma, \delta_1) \subseteq N(\sigma, \delta_2)) \\ \forall \varepsilon_1, \varepsilon_2 \in (0, +\infty) : (\varepsilon_1 < \varepsilon_2 \Rightarrow I(L, \varepsilon_1) \subseteq I(L, \varepsilon_2)) \end{array} \right.$$

thus, decreasing δ or ε tends to make the neighborhood $N(\sigma, \delta)$ or interval $I(L, \varepsilon)$ tighter.

b) We can also show that

$$\left\{ \begin{array}{l} \forall \delta_1, \delta_2 \in (0, +\infty) : N(\sigma, \delta_1) \cap N(\sigma, \delta_2) = N(\sigma, \min\{\delta_1, \delta_2\}) \\ \forall \varepsilon_1, \varepsilon_2 \in (0, +\infty) : I(L, \varepsilon_1) \cap I(L, \varepsilon_2) = I(L, \min\{\varepsilon_1, \varepsilon_2\}) \end{array} \right.$$

c) Relationship between neighborhoods and intervals:

$$\left\{ \begin{array}{l} N(x_0, \delta) = I(x_0, \delta) - \{x_0\} \\ N(+\infty, \delta) = I(+\infty, \delta) \\ N(-\infty, \delta) = I(-\infty, \delta) \end{array} \right.$$

► We now give the following definitions:

Def: Let $A \subseteq \mathbb{R}$ be a set. We say that

$$\sigma \text{ limit point of } A \Leftrightarrow \forall \delta \in (0, +\infty) : N(\sigma, \delta) \cap A \neq \emptyset$$

► interpretation: σ is a limit point of A if and only if regardless of how much we "squeeze" $N(\sigma, \delta)$ by decreasing δ , it always overlaps with A .

Def: Let $f: A \rightarrow \mathbb{R}$ be a function, σ a limit point of A , and $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then

$$\lim_{x \rightarrow \sigma} f(x) = L \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(\sigma, \delta) \Rightarrow f(x) \in I(L, \varepsilon))$$

► interpretation: In the above definition:

ε = how close we want $f(x)$ to be to L

δ = how close x must be brought to σ so that $f(x)$ will be as close to L as required by our choice of ε .

Thus, as we choose smaller ε , it should always be possible to find a smaller δ that works.

► For each choice of σ and L we get a corresponding Weierstrass limit definition (5 choices for σ , 3 choices for L , thus 15 possible definitions). For example, for

$\sigma = x_0 \in \mathbb{R}$ and $L = l \in \mathbb{R}$, we have:

$$\lim_{x \rightarrow x_0} f(x) = l \Leftrightarrow \forall \varepsilon \in (0, +\infty): \exists \delta \in (0, +\infty): \forall x \in A: (0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon)$$

► Note the following immediate consequences of the limit definition

$$\begin{aligned} \lim_{x \rightarrow \sigma} (f(x) - l) = 0 &\Leftrightarrow \lim_{x \rightarrow \sigma} f(x) = l \\ \lim_{x \rightarrow \sigma} f(x) = l &\Leftrightarrow \lim_{x \rightarrow \sigma} [-f(x)] = -l \\ \lim_{x \rightarrow \sigma} f(x) = \pm\infty &\Leftrightarrow \lim_{x \rightarrow \sigma} [-f(x)] = \mp\infty \end{aligned}$$

Def: Let $f: A \rightarrow \mathbb{R}$ be a function, σ a limit point of A .

We say that:

$$\lim_{x \rightarrow \sigma} f(x) \text{ does not exist} \Leftrightarrow \begin{cases} \forall l \in \mathbb{R}: \lim_{x \rightarrow \sigma} f(x) \neq l \\ \lim_{x \rightarrow \sigma} f(x) \neq +\infty \wedge \lim_{x \rightarrow \sigma} f(x) \neq -\infty \end{cases}$$

EXAMPLES

a) Use the limit definition to show that

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 2x - 1}{x^2 + 2x + 9} = 3$$

Solution

Define $\forall x \in \mathbb{R}: f(x) = \frac{3x^2 + 2x - 1}{x^2 + 2x + 9}$, and note that.

$$\begin{aligned} f(x) - 3 &= \frac{3x^2 + 2x - 1}{x^2 + 2x + 9} - 3 = \frac{(3x^2 + 2x - 1) - 3(x^2 + 2x + 9)}{x^2 + 2x + 9} \\ &= \frac{3x^2 + 2x - 1 - 3x^2 - 6x - 27}{x^2 + 2x + 9} = \\ &= \frac{(3-3)x^2 + (2-6)x + (-1-27)}{x^2 + 2x + 9} = \\ &= \frac{-4x - 28}{x^2 + 2x + 9} = \frac{-2(2x + 9)}{x^2 + 2x + 9} \end{aligned}$$

$$\text{with } \begin{cases} \forall x \in \mathbb{R}: x^2 + 2x + 9 = (x+1)^2 + 8 > 0 \\ \forall x \in (-\infty, -5): 2x + 9 < 0 \end{cases}$$

Restrict the domain of f to $A = (-\infty, -5)$.

Let $\varepsilon \in (0, +\infty)$ be given. We have; for all $x \in (-\infty, -5)$:

$$\begin{aligned} |f(x) - 3| &= \left| \frac{-2(2x + 9)}{x^2 + 2x + 9} \right| = \frac{-2(2x + 9)}{x^2 + 2x + 9} \leq \frac{-4x}{x^2 + 2x + 9} \\ &\leq \frac{-4x}{x^2 + 2x} = \frac{-4x}{x(x+2)} = \frac{-4}{x+2} < \varepsilon \Leftrightarrow \end{aligned}$$

$$\Leftrightarrow -4 > \varepsilon(x+2) \quad [\text{via } x+2 < 0]$$

$$\Leftrightarrow \varepsilon x + 2\varepsilon < -4 \Leftrightarrow \varepsilon x < -4 - 2\varepsilon \Leftrightarrow x < \frac{-4 - 2\varepsilon}{\varepsilon}$$

Choose $\delta = \varepsilon / (2\varepsilon + 4) > 0$. Let $x \in A$ be given and assume that $x \in N(-\infty, \delta)$. Then, we have:

$$\begin{aligned} x \in N(-\infty, \delta) &\Rightarrow x \in (-\infty, -1/\delta) \Rightarrow x < -1/\delta \Rightarrow \\ &\Rightarrow x < \frac{-4 - 2\varepsilon}{\varepsilon} \Rightarrow \underline{|f(x) - 3| < \varepsilon} \end{aligned}$$

We have thus shown that

$$\begin{aligned} &\forall \varepsilon \in (0, +\infty): \exists \delta \in (0, +\infty): \forall x \in A: (x \in N(-\infty, \delta) \Rightarrow |f(x) - 3| < \varepsilon) \\ &\Rightarrow \lim_{x \rightarrow -\infty} f(x) = 3 \end{aligned}$$

b) Use the limit definition to show that

$$\lim_{x \rightarrow 1} (x^2 + 2x + 3) = 6$$

Solution

Define $f(x) = x^2 + 2x + 3$, $\forall x \in \mathbb{R}$. Then, we have:

$$\begin{aligned} f(x) - 6 &= (x^2 + 2x + 3) - 6 = x^2 + 2x + 3 - 6 = x^2 + 2x - 3 \\ &= (x+3)(x-1), \forall x \in \mathbb{R}. \end{aligned}$$

Restrict the domain of f to $A = (0, 1) \cup (1, 2)$. Let

$\varepsilon \in (0, +\infty)$ be given. Then, for all $x \in A$, we have:

$$\begin{aligned} |f(x) - 6| &= |(x+3)(x-1)| = |x+3||x-1| = |(x-1)+4||x-1| \\ &\leq [|x-1| + 4]|x-1| < [1+4]|x-1| = 5|x-1| \\ &= 5|x-1| < \varepsilon \iff |x-1| < \varepsilon/5. \end{aligned}$$

Choose $\delta = \varepsilon/5$. Let $x \in A$ be given and assume that $x \in N(1, \delta)$. Then, we have:

$$\begin{aligned} x \in N(1, \delta) &\Rightarrow 0 < |x-1| < \delta \Rightarrow 0 < |x-1| < \varepsilon/5 \Rightarrow \\ &\Rightarrow |f(x) - 6| < \varepsilon. \end{aligned}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(1, \delta) \Rightarrow |f(x) - 6| < \varepsilon)$$

$$\Rightarrow \lim_{x \rightarrow 1} (x^2 + 2x + 3) = 6$$

THEORY QUESTIONS

① State the definition of the neighborhood $N(\sigma, \delta)$ for an arbitrary σ and the definition of the statement: σ limit point of A with $A \subseteq \mathbb{R}$, using quantifier notation.

② State the definition of the neighborhood $N(\sigma, \delta)$ and interval $I(L, \epsilon)$ for an arbitrary choice of σ and L , and then state the general definition of the statement $\lim_{x \rightarrow \sigma} f(x) = L$, using quantifier notation.

③ State the definition of the statement $\lim_{x \rightarrow \sigma} f(x)$ does not exist

for an arbitrary choice of σ .

④ Use quantifiers to write the specific definitions for the following statements, without using the neighborhood / interval notation

a) $\lim_{x \rightarrow x_0} f(x) = l \in \mathbb{R}$

b) $\lim_{x \rightarrow x_0^-} f(x) = -\infty$

c) $\lim_{x \rightarrow x_0^+} f(x) = +\infty$

d) $\lim_{x \rightarrow -\infty} f(x) = l \in \mathbb{R}$

e) $\lim_{x \rightarrow +\infty} f(x) = -\infty$

f) $\lim_{x \rightarrow -\infty} f(x) = +\infty$

$$g) \lim_{x \rightarrow x_0^+} f(x) = l \in \mathbb{R}$$

$$h) \lim_{x \rightarrow 0} f(x) = 0$$

EXERCISES

⑤ Use the neighborhood definition and proof by cases to show that

$$a) \forall \delta_1, \delta_2 \in (0, +\infty): (\delta_1 < \delta_2 \Rightarrow N(\sigma, \delta_1) \subseteq N(\sigma, \delta_2))$$

$$b) \forall \delta_1, \delta_2 \in (0, +\infty): N(\sigma, \delta_1) \cap N(\sigma, \delta_2) = N(\sigma, \min\{\delta_1, \delta_2\})$$

↑ Use (a) to prove (b).

⑥ Use the limit definition to show that:

$$a) \lim_{x \rightarrow -\infty} \frac{2x-3}{x-5} = 2$$

$$b) \lim_{x \rightarrow +\infty} \frac{x^2}{x-2} = +\infty$$

$$c) \lim_{x \rightarrow +\infty} \frac{\sin(3x)}{x+1} = 0$$

$$d) \lim_{x \rightarrow +\infty} \frac{2x^2+3x+1}{3x^2+x+5} = \frac{2}{3}$$

$$e) \lim_{x \rightarrow 1} \frac{(x+1)(x+3)}{x^2+2x+5} = 1$$

$$f) \lim_{x \rightarrow 3^-} \frac{x^3}{2x+1} = \frac{27}{7}$$

$$g) \lim_{x \rightarrow 2^+} \frac{3x+1}{x-2} = +\infty$$

$$h) \lim_{x \rightarrow 3^-} \frac{5x+2}{x-3} = -\infty$$

$$i) \lim_{x \rightarrow -1^+} \frac{x^3}{2x+2} = -\infty$$

$$j) \lim_{x \rightarrow -\infty} \frac{2x \cos x}{x^2-2} = 0$$

$$k) \lim_{x \rightarrow +\infty} \frac{3x^2 \cos x}{x^3+2} = 0$$

$$l) \lim_{x \rightarrow 0} x^3 \sin(1/x) = 0$$

$$m) \lim_{x \rightarrow +\infty} \frac{\cos x}{x^3} = 0$$

⑦ Use the limit definition to show that

$$a) \lim_{x \rightarrow -\infty} \frac{2x \cos x}{x^2 - 2} = 0$$

$$b) \lim_{x \rightarrow +\infty} \frac{3x^2 \cos x}{x^3 + 2} = 0$$

$$c) \lim_{x \rightarrow 0} x^3 \sin(1/x) = 0$$

$$d) \lim_{x \rightarrow +\infty} \frac{\cos x}{x^3} = 0$$

→ Relation between side limits $x \rightarrow x_0^+$ and $x \rightarrow x_0^-$

Thm: Let $f: A \rightarrow \mathbb{R}$ be a function, let x_0 be a limit point of A , and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then, we have

$$\lim_{x \rightarrow x_0} f(x) = L \iff \lim_{x \rightarrow x_0^+} f(x) = L \wedge \lim_{x \rightarrow x_0^-} f(x) = L$$

Proof

$(\Rightarrow)$: Assume that $\lim_{x \rightarrow x_0} f(x) = L$. Then, we have:

$$\begin{aligned} \lim_{x \rightarrow x_0} f(x) = L &\Rightarrow \\ &\Rightarrow \forall \varepsilon \in (0, +\infty): \exists \delta \in (0, +\infty): \forall x \in A: (x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)) \\ &\quad \Rightarrow f(x) \in I(L, \varepsilon) \end{aligned}$$

Let $\varepsilon \in (0, +\infty)$ be given. Choose $\delta \in (0, +\infty)$ such that

$$\forall x \in A: (x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)) \Rightarrow f(x) \in I(L, \varepsilon)$$

Let $x \in A$ be given. Then, we have:

$$\begin{aligned} \underline{x \in (x_0, x_0 + \delta)} &\Rightarrow x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \Rightarrow \\ &\Rightarrow \underline{f(x) \in I(L, \varepsilon)} \end{aligned}$$

and

$$\begin{aligned} \underline{x \in (x_0 - \delta, x_0)} &\Rightarrow x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \Rightarrow \\ &\Rightarrow \underline{f(x) \in I(L, \varepsilon)} \end{aligned}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty): \exists \delta \in (0, +\infty): \forall x \in A: \begin{cases} x \in (x_0 - \delta, x_0) \Rightarrow f(x) \in I(L, \varepsilon) \\ x \in (x_0, x_0 + \delta) \Rightarrow f(x) \in I(L, \varepsilon) \end{cases}$$

$$\Rightarrow \lim_{x \rightarrow x_0^+} f(x) = L \wedge \lim_{x \rightarrow x_0^-} f(x) = L.$$

($\Rightarrow$): Assume that $\lim_{x \rightarrow x_0^+} f(x) = L \wedge \lim_{x \rightarrow x_0^-} f(x) = L$.

Let $\varepsilon \in (0, +\infty)$ be given. Then, we have:

$$\left\{ \begin{array}{l} \lim_{x \rightarrow x_0^+} f(x) = L \\ \lim_{x \rightarrow x_0^-} f(x) = L \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \exists \delta \in (0, +\infty) : \forall x \in A : (x \in (x_0 - \delta, x_0) \Rightarrow f(x) \in I(L, \varepsilon)) \\ \exists \delta \in (0, +\infty) : \forall x \in A : (x \in (x_0, x_0 + \delta) \Rightarrow f(x) \in I(L, \varepsilon)) \end{array} \right.$$

Choose $\delta_1, \delta_2 \in (0, +\infty)$ such that

$$\left\{ \begin{array}{l} \forall x \in A : (x \in (x_0 - \delta_1, x_0) \Rightarrow f(x) \in I(L, \varepsilon)) \\ \forall x \in A : (x \in (x_0, x_0 + \delta_2) \Rightarrow f(x) \in I(L, \varepsilon)) \end{array} \right.$$

Choose $\delta = \min \{\delta_1, \delta_2\}$. Let $x \in A$ be given and

assume that $x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$. Then, we have:

$$\begin{aligned} x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) &\Rightarrow x \in (x_0 - \delta, x_0) \vee x \in (x_0, x_0 + \delta) \\ &\Rightarrow x \in (x_0 - \delta_1, x_0) \vee x \in (x_0, x_0 + \delta_2) \\ &\Rightarrow f(x) \in I(L, \varepsilon) \vee f(x) \in I(L, \varepsilon) \\ &\Rightarrow \underline{f(x) \in I(L, \varepsilon)} \end{aligned}$$

We have thus shown that

$$\begin{aligned} \forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \Rightarrow \\ \Rightarrow f(x) \in I(L, \varepsilon)) \end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow x_0} f(x) = L$$

□

► An immediate consequence of this result is the following statement:

$\lim_{x \rightarrow x_0^+} f(x) = L_1 \wedge \lim_{x \rightarrow x_0^-} f(x) = L_2 \wedge L_1 \neq L_2 \Rightarrow \lim_{x \rightarrow x_0} f(x) \text{ does not exist}$

• Methodology: To show that $\lim_{x \rightarrow \sigma} f(x) = L$ by definition

•₁ Investigate $f(x) \in I(L, \varepsilon)$ and, if needed, restrict the domain A of f to $A = A_0 \cap N(\sigma, \delta)$ for an appropriate δ with A_0 the widest possible domain.

•₂ Let $\varepsilon \in (0, +\infty)$ be given. Derive an equivalence $f(x) \in \mathcal{S} \subseteq I(L, \varepsilon) \Leftrightarrow x \in N(\sigma, g(\varepsilon))$

•₃ Choose $\delta = g(\varepsilon)$. Let $x \in A$ be given and assume that $x \in N(\sigma, \delta)$. Then, we have $x \in N(\sigma, \delta) \Rightarrow x \in N(\sigma, g(\varepsilon)) \Rightarrow f(x) \in \mathcal{S} \Rightarrow f(x) \in I(L, \varepsilon)$.

•₄ We have thus shown that

$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(\sigma, \delta) \Rightarrow f(x) \in I(L, \varepsilon))$
 $\Rightarrow \lim_{x \rightarrow \sigma} f(x) = L.$

EXAMPLES

a) Use the limit definition to show that for

$$f(x) = \begin{cases} 3x & , \text{ if } x \in [1, +\infty) \\ -x+4 & , \text{ if } x \in (-\infty, 1) \end{cases}$$

we have $\lim_{x \rightarrow 1} f(x) = 3$

Solution

• Limit $x \rightarrow 1^+$: Restrict the domain of f to $A = [1, +\infty)$.

Let $\varepsilon \in (0, +\infty)$ be given. Then for all $x \in A$, we have

$$|f(x) - 3| = |3x - 3| = |3(x-1)| = 3|x-1| < \varepsilon \Leftrightarrow |x-1| < \varepsilon/3.$$

Choose $\delta = \varepsilon/3$. Let $x \in A$ be given and assume that $x \in N(1^+, \delta)$. Then, we have:

$$x \in N(1^+, \delta) \Rightarrow x \in N(1, \delta) \Rightarrow 0 < |x-1| < \delta \Rightarrow$$

$$\Rightarrow |x-1| < \varepsilon/3 \Rightarrow \underline{|f(x) - 3| < \varepsilon}.$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(1^+, \delta) \Rightarrow |f(x) - 3| < \varepsilon)$$

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = 3.$$

• Limit $x \rightarrow 1^-$: Restrict the domain of f to $A = (-\infty, 1)$.

Let $\varepsilon \in (0, +\infty)$ be given. Then, for all $x \in A$, we have

$$|f(x) - 3| = |(-x+4) - 3| = |-x+1| = |x-1|.$$

Choose $\delta = \varepsilon$. Let $x \in A$ be given and assume that $x \in N(1^-, \delta)$. Then, we have:

$$x \in N(1^-, \delta) \Rightarrow x \in N(1, \delta) \Rightarrow 0 < |x-1| < \delta \Rightarrow$$

$$\Rightarrow |x-1| < \varepsilon \Rightarrow \underline{|f(x) - 3| < \varepsilon}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty): \exists \delta \in (0, +\infty): \forall x \in A: (x \in N(1, \delta) \Rightarrow |f(x) - 3| < \varepsilon) \\ \Rightarrow \lim_{x \rightarrow 1^-} f(x) = 3.$$

• From the above, we conclude that

$$\lim_{x \rightarrow 1^+} f(x) = 3 \wedge \lim_{x \rightarrow 1^-} f(x) = 3 \Rightarrow \lim_{x \rightarrow 1} f(x) = 3.$$

THEORY QUESTIONS

(8) Let $f: A \rightarrow \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be a limit point of A and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Show that

$$a) \lim_{x \rightarrow x_0} f(x) = L \Rightarrow \left(\lim_{x \rightarrow x_0^+} f(x) = L \wedge \lim_{x \rightarrow x_0^-} f(x) = L \right)$$

$$b) \left(\lim_{x \rightarrow x_0^+} f(x) = L \wedge \lim_{x \rightarrow x_0^-} f(x) = L \right) \Rightarrow \lim_{x \rightarrow x_0} f(x) = L$$

EXERCISES

(9) Let $f: A \rightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}$ limit point of A and let $L_1, L_2 \in \mathbb{R} \cup \{+\infty, -\infty\}$. Show that

$$\left\{ \begin{array}{l} \lim_{x \rightarrow x_0^+} f(x) = L_1 \\ \lim_{x \rightarrow x_0^-} f(x) = L_2 \\ L_1 \neq L_2 \end{array} \right\} \Rightarrow \lim_{x \rightarrow x_0} f(x) \text{ does not exist.}$$

(10) Use the limit definition, in conjunction with side limits, to show that

$$a) \lim_{x \rightarrow 2} f(x) = 7 \quad \text{with } f(x) = \begin{cases} 3x+1, & \text{if } x \in [2, +\infty) \\ 4x-1, & \text{if } x \in (-\infty, 2) \end{cases}$$

$$b) \lim_{x \rightarrow 1} f(x) = 2 \quad \text{with } f(x) = \begin{cases} x^2+x, & \text{if } x \in (-\infty, 1) \\ 2x^3, & \text{if } x \in (1, +\infty) \end{cases}$$

c) $\lim_{x \rightarrow -1} f(x)$ does not exist, with

$$f(x) = \begin{cases} 3x^2, & \text{if } x \in (-\infty, -1) \\ 3x, & \text{if } x \in (-1, +\infty) \end{cases}$$

d) $\lim_{x \rightarrow 3} f(x)$ does not exist, with

$$f(x) = \begin{cases} x(x+2), & \text{if } x \in (3, +\infty) \\ x^2 - 2, & \text{if } x \in (-\infty, 3). \end{cases}$$

▼ Function limits as net limits

Function limits are a special case of a net limit, and as such they inherit all the properties that we have previously established on convergent nets. The connection between the two concepts is established by the following theorem:

Thm: Let $f: A \rightarrow \mathbb{R}$ be a function, let σ be a limit point of A , let $\delta_0 \in (0, +\infty)$, and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then, define $(D, <_\sigma)$ such that

$$\begin{cases} D = N(\sigma, \delta_0) \cap A \\ \|x\|_\sigma = \inf \{ \delta \in (0, +\infty) \mid x \in N(\sigma, \delta) \cap A \} \\ \forall x_1, x_2 \in D : (x_1 <_\sigma x_2 \Leftrightarrow \|x_1\|_\sigma \geq \|x_2\|_\sigma) \end{cases}$$

Then, we have:

$$\begin{cases} (D, <_\sigma) \text{ is a directed set} \\ \lim_{x \rightarrow \sigma} f(x) = L \Leftrightarrow \lim_{x \in D} f(x) = L \end{cases}$$

→ Note that $\|x\|_\sigma$ represents how close x is to the limit point σ . Also, $x_1 <_\sigma x_2$ is the statement that x_2 is closer to the limit point σ than x_1 .

Proof

Since σ limit point of A , it follows that $D = N(\sigma, \delta_0) \cap A \neq \emptyset$.

Define $\phi(x) = \{ \delta \in (0, +\infty) \mid x \in N(\sigma, \delta) \cap A \}$.

► We claim that $\|x\|_\sigma = \inf \phi(x)$ is well-defined.
for all $x \in D$.

Let $x \in D$ be given. Then, we have:

$$S(x) = \{\delta \in (0, +\infty) \mid x \in N(\sigma, \delta)\} \subseteq (0, +\infty) \subseteq \mathbb{R} \Rightarrow S(x) \subseteq \mathbb{R} \quad (1)$$

and

$$x \in D \Rightarrow x \in N(\sigma, S_0) \cap A \Rightarrow S_0 \in S(x) \Rightarrow S(x) \neq \emptyset \quad (2)$$

and

$$S(x) \subseteq (0, +\infty) \Rightarrow \forall \delta \in S(x) : \delta \in (0, +\infty)$$

$$\Rightarrow \forall \delta \in S(x) : \delta \geq 0$$

$$\Rightarrow S(x) \text{ lower bounded} \quad (3)$$

From Eq. (1), Eq. (2), Eq. (3) via the axiom of completeness it follows that $\|x\|_\sigma = \inf S(x)$ is well-defined.

This proves the claim

► We will show that $(D, <_\sigma)$ is a directed set.

• $<_\sigma$ reflective property.

Let $x \in D$ be given. Then $\|x\|_\sigma \geq \|x\|_\sigma \Rightarrow \underline{x <_\sigma x}$. It follows that:

$$\forall x \in D : x <_\sigma x$$

• $<_\sigma$ transitive property.

Let $x, y, z \in D$ be given and assume that $\underline{x <_\sigma y}$ and $\underline{y <_\sigma z}$. Then, we have:

$$\begin{cases} x <_\sigma y \\ y <_\sigma z \end{cases} \Rightarrow \begin{cases} \|x\|_\sigma \geq \|y\|_\sigma \\ \|y\|_\sigma \geq \|z\|_\sigma \end{cases} \Rightarrow \|x\|_\sigma \geq \|z\|_\sigma \Rightarrow \underline{x <_\sigma z}$$

We have thus shown that

$$\forall x, y, z \in D : ((x <_\sigma y \wedge y <_\sigma z) \Rightarrow x <_\sigma z)$$

• $<_\sigma$ refinement property

Let $\underline{x, y \in D}$ be given. Since σ is a limit point of A ,

choose $z \in N(\sigma, \min\{\|x\|_\sigma, \|y\|_\sigma\}) \cap A$. Then, we have:
 $\|z\|_\sigma \leq \min\{\|x\|_\sigma, \|y\|_\sigma\} \Rightarrow \begin{cases} \|z\|_\sigma \leq \|x\|_\sigma \\ \|z\|_\sigma \leq \|y\|_\sigma \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} x <_\sigma z \\ y <_\sigma z \end{cases}$$

We have thus shown that

$$\forall x, y \in D : \exists z \in D : (x <_\sigma z \wedge y <_\sigma z)$$

From the above, we conclude that $(D, <_\sigma)$ is a directed set.

► We will show that $\lim_{x \rightarrow \sigma} f(x) = L \Leftrightarrow \lim_{x \in D} f(x) = L$.

($\Rightarrow$): Assume that $\lim_{x \rightarrow \sigma} f(x) = L$. It follows that

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(\sigma, \delta) \Rightarrow f(x) \in I(L, \varepsilon))$$

Let $\varepsilon \in (0, +\infty)$ be given. Choose $\delta \in (0, +\infty)$ such that

$$\forall x \in A : (x \in N(\sigma, \delta) \Rightarrow f(x) \in I(L, \varepsilon))$$

Since σ limit point of A , we can choose

$$n_0 \in N(\sigma, \min\{\delta_0, \delta\}) \cap A \subseteq D. \Rightarrow \underline{n_0 \in D.}$$

and note that $\|n_0\|_\sigma \leq \min\{\delta_0, \delta\}$. Let $\underline{n \in D}$ be given

and assume that $\underline{n >_\sigma n_0}$. Then, we have:

$$n >_\sigma n_0 \Rightarrow \|n\|_\sigma \leq \|n_0\|_\sigma \leq \min\{\delta_0, \delta\} \leq \delta \Rightarrow$$

$$\Rightarrow \|n\|_\sigma \leq \delta \Rightarrow n \in N(\sigma, \delta) \cap A \Rightarrow \underline{f(n) \in I(L, \varepsilon)}.$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n >_\sigma n_0 \Rightarrow f(n) \in I(L, \varepsilon))$$

$$\Rightarrow \lim_{n \in D} f(n) = L.$$

($\Leftarrow$): Assume that $\lim_{n \in D} f(n) = L$. Then, we have:

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{D} : \forall n \in \mathbb{D} : (n >_\sigma n_0 \Rightarrow f(n) \in I(L, \varepsilon))$$

Let $\varepsilon \in (0, +\infty)$ be given. Choose $n_0 \in \mathbb{D}$ such that

$$\forall n \in \mathbb{D} : (n >_\sigma n_0 \Rightarrow f(n) \in I(L, \varepsilon))$$

Choose $\delta = \|n_0\|_\sigma \in (0, +\infty)$. Let $x \in A$ be given and assume that $x \in N(\sigma, \delta)$. Then, we have:

$$\begin{cases} x \in A \\ x \in N(\sigma, \delta) \end{cases} \Rightarrow x \in N(\sigma, \delta) \cap A \Rightarrow \|x\|_\sigma \leq \delta = \|n_0\|_\sigma$$

$$\Rightarrow \|x\|_\sigma \leq \|n_0\|_\sigma \Rightarrow x >_\sigma n_0 \Rightarrow \underline{f(x) \in I(L, \varepsilon)}.$$

We have thus shown that:

$$\begin{aligned} &\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(\sigma, \delta) \Rightarrow f(x) \in I(L, \varepsilon)) \\ &\Rightarrow \lim_{x \rightarrow \sigma} f(x) = L. \end{aligned}$$

This concludes the proof

□

▼ Properties of limits of functions

Since limits of functions are special cases of net limits, the following properties of function limits are immediately obtained:

① → Uniqueness

Let $f: A \rightarrow \mathbb{R}$ with σ limit point of A and let $L_1, L_2 \in \mathbb{R} \cup \{\pm\infty\}$. Then, we have:

$$\left(\lim_{x \rightarrow \sigma} f(x) = L_1 \wedge \lim_{x \rightarrow \sigma} f(x) = L_2 \right) \Rightarrow L_1 = L_2.$$

② → Functions with finite limits

Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ and let σ be a limit point of both A and B and assume that:

$$\lim_{x \rightarrow \sigma} f(x) = l_1 \in \mathbb{R} \wedge \lim_{x \rightarrow \sigma} g(x) = l_2 \in \mathbb{R}$$

Then, we have:

$$a) \lim_{x \rightarrow \sigma} [f(x) + g(x)] = l_1 + l_2$$

$$c) \lim_{x \rightarrow \sigma} |f(x)| = |l_1|$$

$$b) \lim_{x \rightarrow \sigma} [f(x) g(x)] = l_1 l_2$$

$$f) l_1 > 0 \Rightarrow \forall k \in \mathbb{N}^*: \lim_{x \rightarrow \sigma} \sqrt[k]{f(x)} = \sqrt[k]{l_1}$$

$$c) \forall a \in \mathbb{R}: \lim_{x \rightarrow \sigma} [af(x)] = al_1$$

$$d) l_2 \neq 0 \Rightarrow \lim_{x \rightarrow \sigma} \left(\frac{f(x)}{g(x)} \right) = \frac{l_1}{l_2}$$

③ → Functions with limits going to infinity.

Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ and let σ be a limit point of A and B . Let $\delta \in (0, +\infty)$ and $a \in \mathbb{R}$. Then, we have:

$$a) \begin{cases} \forall x \in N(\sigma, \delta) \cap B: g(x) > a \\ \lim_{x \rightarrow \sigma} f(x) = +\infty \end{cases} \Rightarrow \lim_{x \rightarrow \sigma} [f(x) + g(x)] = +\infty$$

$$b) \begin{cases} \forall x \in N(\sigma, \delta) \cap B: g(x) < a \\ \lim_{x \rightarrow \sigma} f(x) = -\infty \end{cases} \Rightarrow \lim_{x \rightarrow \sigma} [f(x) + g(x)] = -\infty$$

$$c) \begin{cases} \forall x \in N(\sigma, \delta) \cap B: g(x) > a > 0 \\ \lim_{x \rightarrow \sigma} f(x) = \pm\infty \end{cases} \Rightarrow \lim_{x \rightarrow \sigma} [f(x)g(x)] = \pm\infty$$

$$d) \begin{cases} \forall x \in N(\sigma, \delta) \cap B: g(x) < a < 0 \\ \lim_{x \rightarrow \sigma} f(x) = \pm\infty \end{cases} \Rightarrow \lim_{x \rightarrow \sigma} [f(x)g(x)] = \mp\infty$$

When the limit of $g(x)$ is also known, then this result can be combined with the following statements:

$$\begin{aligned} \lim_{x \rightarrow \sigma} g(x) > 0 \vee \lim_{x \rightarrow \sigma} g(x) = +\infty &\Rightarrow \exists \delta \in (0, +\infty): \exists a \in \mathbb{R}: \forall x \in N(\sigma, \delta) \cap B: \\ &\quad : g(x) > a > 0 \\ \lim_{x \rightarrow \sigma} g(x) < 0 \vee \lim_{x \rightarrow \sigma} g(x) = -\infty &\Rightarrow \exists \delta \in (0, +\infty): \exists a \in \mathbb{R}: \forall x \in N(\sigma, \delta) \cap B: \\ &\quad : g(x) < a < 0 \end{aligned}$$

giving several deductions that are summarized in the tables given below:

$f(x) \downarrow g(x) \rightarrow$	a	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	$?$
$-\infty$	$-\infty$	$?$	$-\infty$

$$\lim_{x \rightarrow \sigma} [f(x) + g(x)]$$

$f(x) \downarrow g(x) \rightarrow$	0	$p > 0$	$n < 0$	$+\infty$	$-\infty$
$+\infty$	$?$	$+\infty$	$-\infty$	$+\infty$	$-\infty$
$-\infty$	$?$	$-\infty$	$+\infty$	$-\infty$	$+\infty$

$$\lim_{x \rightarrow \sigma} [f(x)g(x)]$$

→ The "?" correspond to indeterminate forms. It means that the limit cannot be determined without more information, and the limit may or may not exist.

④ → Limit forms $K/0$, $1/(\pm\infty)$.

Let $f: A \rightarrow \mathbb{R}$ and $\delta \in (0, +\infty)$ and let σ be a limit point of A .

$$a) \begin{cases} \forall x \in N(\sigma, \delta) \cap A: f(x) > 0 \\ \lim_{x \rightarrow \sigma} f(x) = 0 \end{cases} \Rightarrow \lim_{x \rightarrow \sigma} \frac{1}{f(x)} = +\infty$$

$$b) \begin{cases} \forall x \in N(\sigma, \delta) \cap A: f(x) < 0 \\ \lim_{x \rightarrow \sigma} f(x) = 0 \end{cases} \Rightarrow \lim_{x \rightarrow \sigma} \frac{1}{f(x)} = -\infty$$

$$c) \lim_{x \rightarrow \sigma} f(x) \in \{+\infty, -\infty\} \Rightarrow \lim_{x \rightarrow \sigma} \frac{1}{f(x)} = 0$$

→ Immediate consequences of limit properties

The following results are immediate consequences of the limit properties

① → Monomial function

$$\forall x_0 \in \mathbb{R} : \forall k \in \mathbb{N}^+ : \lim_{x \rightarrow x_0} x^k = x_0^k$$

$$\forall k \in \mathbb{N}^+ : \lim_{x \rightarrow +\infty} x^k = +\infty$$

$$\forall k \in \mathbb{N}^+ : \lim_{x \rightarrow -\infty} x^{2k+1} = -\infty$$

$$\forall k \in \mathbb{N}^+ : \lim_{x \rightarrow -\infty} x^{2k} = +\infty$$

$$\forall k \in \mathbb{N}^+ : \lim_{x \rightarrow \pm\infty} x^{-k} = 0$$

② → Polynomial function

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$.

Then, we have:

$$a) \forall x_0 \in \mathbb{R} : \lim_{x \rightarrow x_0} f(x) = f(x_0)$$

$$b) \lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} a_n x^n$$

③ → Rational function

Let $P: \mathbb{R} \rightarrow \mathbb{R}$ and $Q: \mathbb{R} \rightarrow \mathbb{R}$ with

$$\begin{cases} P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \\ Q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0 \end{cases}$$

Then, we have:

$$a) \forall x_0 \in \mathbb{R}: (Q(x_0) \neq 0 \Rightarrow \lim_{x \rightarrow x_0} \frac{P(x)}{Q(x)} = \frac{P(x_0)}{Q(x_0)})$$

$$b) \lim_{x \rightarrow \pm\infty} \frac{P(x)}{Q(x)} = \lim_{x \rightarrow \pm\infty} \frac{a_n x^n}{b_m x^m}$$

④ → Rational k/o limits

$$\forall a \in \mathbb{R}: \lim_{x \rightarrow a^+} \frac{1}{x-a} = +\infty$$

$$\forall a \in \mathbb{R}: \lim_{x \rightarrow a^-} \frac{1}{x-a} = -\infty$$

$$\forall a \in \mathbb{R}: \lim_{x \rightarrow a} \frac{1}{x-a} = +\infty$$

EXAMPLES

a) Let $f(x) = \frac{ax^3 + x - 2}{(a-1)x^2 + x + 1}$. Use the properties of limits

to calculate $\lim_{x \rightarrow +\infty} f(x)$

Solution

We distinguish between the following cases:

Case 1: Assume that $a \in \mathbb{R} - \{0, 1\}$. Then, we have:

$$\begin{aligned} \lim_{x \rightarrow +\infty} f(x) &= \lim_{x \rightarrow +\infty} \frac{ax^3 + x - 2}{(a-1)x^2 + x + 1} = \lim_{x \rightarrow +\infty} \frac{ax^3}{(a-1)x^2} = \\ &= \frac{a}{a-1} \lim_{x \rightarrow +\infty} x = \frac{a}{a-1} (+\infty) \end{aligned}$$

We use a sign table for the sign of $a/(a-1)$:

a	0	1
a	-	+
a-1	-	+
	+	-

to conclude that:

$$\lim_{x \rightarrow +\infty} f(x) = \begin{cases} +\infty, & \text{if } a \in (-\infty, 0) \cup (1, +\infty) \\ -\infty, & \text{if } a \in (0, 1) \end{cases}$$

Case 2: Assume that $a = 0$. Then, we have:

$$\begin{aligned} \lim_{x \rightarrow +\infty} f(x) &= \lim_{x \rightarrow +\infty} \frac{x - 2}{-x^2 + x + 1} = \lim_{x \rightarrow +\infty} \frac{x}{-x^2} = \\ &= \lim_{x \rightarrow +\infty} \frac{-1}{x} = 0 \end{aligned}$$

Case 3: Assume that $a=1$. Then, we have:

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{x^3 + x - 2}{x + 1} = \lim_{x \rightarrow +\infty} \frac{x^3}{x} = \lim_{x \rightarrow +\infty} x^2 = +\infty.$$

We conclude that:

$$\lim_{x \rightarrow +\infty} f(x) = \begin{cases} +\infty, & \text{if } a \in (-\infty, 0) \cup [1, +\infty) \\ -\infty, & \text{if } a \in (0, 1) \\ 0, & \text{if } a = 0 \end{cases}$$

b) Use the limit properties to calculate $\lim_{x \rightarrow -\infty} f(x)$ for $f(x) = \sqrt{x^2 - 5x + 6} + ax$, for all $a \in \mathbb{R}$.

Solution

Since

$$\lim_{x \rightarrow -\infty} (x^2 - 5x + 6) = \lim_{x \rightarrow -\infty} x^2 = +\infty \Rightarrow$$

$$\Rightarrow \exists \mu \in (0, +\infty) : \forall x \in (-\infty, -\mu) : x^2 - 5x + 6 > 0$$

it follows that $-\infty$ is a limit point of the domain of f for all $a \in \mathbb{R}$. Choose $\mu \in (0, +\infty)$ such that $\forall x \in (-\infty, -\mu) : x^2 - 5x + 6 > 0$.

Then, we have:

$$\begin{aligned} \forall x \in (-\infty, -\mu) : f(x) &= \sqrt{x^2 - 5x + 6} + ax = \\ &= \sqrt{x^2(1 - 5x^{-1} + 6x^{-2})} + ax = \\ &= |x| \sqrt{1 - 5x^{-1} + 6x^{-2}} + ax = \\ &= -x \sqrt{1 - 5x^{-1} + 6x^{-2}} + ax = \quad [\text{via } x < -\mu < 0] \\ &= x(a - \sqrt{1 - 5x^{-1} + 6x^{-2}}) \equiv xg(x) \end{aligned}$$

where we define $g(x) = a - \sqrt{1 - 5x^{-1} + 6x^{-2}}$ and

$$\begin{aligned} \lim_{x \rightarrow -\infty} g(x) &= \lim_{x \rightarrow -\infty} [a - \sqrt{1 - 5x^{-1} + 6x^{-2}}] = a - \sqrt{1 - 0 + 0} \\ &= a - 1 \end{aligned}$$

We distinguish between the following cases:

Case 1: Assume that $a \in (1, +\infty)$. Then, we have:

$$\lim_{x \rightarrow -\infty} x = -\infty \wedge \lim_{x \rightarrow -\infty} g(x) = a - 1 > 0 \Rightarrow \lim_{x \rightarrow -\infty} f(x) = -\infty$$

Case 2: Assume that $a \in (-\infty, 1)$. Then, we have:

$$\lim_{x \rightarrow -\infty} x = -\infty \wedge \lim_{x \rightarrow -\infty} g(x) = a - 1 < 0 \Rightarrow \lim_{x \rightarrow +\infty} f(x) = +\infty$$

Case 3: Assume that $a = 1$. Then, we have:

$$\begin{aligned} f(x) &= \sqrt{x^2 - 5x + 6} + x = \frac{(\sqrt{x^2 - 5x + 6})^2 - x^2}{\sqrt{x^2 - 5x + 6} - x} = \\ &= \frac{x^2 - 5x + 6 - x^2}{\sqrt{x^2 - 5x + 6} - x} = \frac{-5x + 6}{\sqrt{x^2 - 5x + 6} - x} \\ &= \frac{-5 + 6x^{-1}}{-\sqrt{1 - 5x^{-1} + 6x^{-2}} - 1}, \quad \forall x \in (-\infty, -p) \end{aligned}$$

$$\begin{aligned} \rightarrow \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{-5 + 6x^{-1}}{-\sqrt{1 - 5x^{-1} + 6x^{-2}} - 1} \\ &= \frac{-5 + 0}{-\sqrt{1 - 0 + 0} - 1} = \frac{-5}{-1 - 1} = \frac{5}{2} \end{aligned}$$

From all of the above, we conclude that

$$\lim_{x \rightarrow -\infty} f(x) = \begin{cases} -\infty, & \text{if } a \in (1, +\infty) \\ +\infty, & \text{if } a \in (-\infty, 1) \\ 5/2, & \text{if } a = 1 \end{cases}$$

THEORY QUESTION

- (11) Let $f: A \rightarrow \mathbb{R}$ and let σ be a limit point of A and let $L \in \mathbb{R} \cup \{\pm\infty\}$. State the construction of the limit statement $\lim_{x \rightarrow \sigma} f(x) = L$ in terms of a directed set $(D, <)$ and the corresponding directed set limits.

EXERCISES

- (12) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\forall x \in \mathbb{R}: f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$\text{Show that: } \lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} a_n x^n$$

- (13) Let $p: \mathbb{R} \rightarrow \mathbb{R}$ and $q: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\forall x \in \mathbb{R}: \begin{cases} p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \\ q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0 \end{cases}$$

$$\text{Show that: } \lim_{x \rightarrow \pm\infty} \frac{p(x)}{q(x)} = \lim_{x \rightarrow \pm\infty} \frac{a_n x^n}{b_m x^m}$$

- (14) Use the limit properties to evaluate the following limits for all $a \in \mathbb{R}$.

$$\text{a) } \lim_{x \rightarrow +\infty} f(x) \text{ with } f(x) = \frac{(a^2 - 1)x^2 + ax}{(a - 1)x^2 + (a + 1)x + 3}$$

$$\text{b) } \lim_{x \rightarrow -\infty} f(x) \text{ with } f(x) = \frac{(a - 1)x^3 + (a + 1)x}{(a + 1)x^3 + (a - 1)x}$$

$$c) \lim_{x \rightarrow +\infty} f(x) \text{ with } f(x) = \frac{ax^3 + (a+1)x^2 - ax + 3}{(a+1)x^2 + 2ax + 1}$$

$$d) \lim_{x \rightarrow -\infty} f(x) \text{ with } f(x) = \frac{x^2}{x+a} - \frac{x^2}{x-a}$$

(15) Consider the function $f: A \rightarrow \mathbb{R}$ with $f(x) = \sqrt{a - x^2 - 2x} - x$ with $a \in \mathbb{R}$. Show that:

$\forall a \in \mathbb{R}: \lim_{x \rightarrow +\infty} f(x)$ not well-defined.

↑ $\rightarrow$ We say that for $f: A \rightarrow \mathbb{R}$, the limit $\lim_{x \rightarrow \infty} f(x)$ is not well-defined $\Leftrightarrow$ or not a limit point of A .

(16) Find the set $S \subseteq \mathbb{R}$ of all $a \in \mathbb{R}$ for which the following limits are well-defined, and then evaluate the limit in terms of the parameter $a \in S$.

$$a) \lim_{x \rightarrow +\infty} f(x) \text{ with } f(x) = x(\sqrt{ax^2 + 6x + 3} - x)$$

$$b) \lim_{x \rightarrow -\infty} f(x) \text{ with } f(x) = \sqrt{ax^2 + 2x - 1} - \sqrt{x^2 + 1}$$

$$c) \lim_{x \rightarrow +\infty} f(x) \text{ with } f(x) = \sqrt{x^2 - 4x + a} - x$$

$$d) \lim_{x \rightarrow -\infty} f(x) \text{ with } f(x) = |x|[\sqrt{ax^2 + 2x + 1} - x]$$

(17) Use the limit properties to show that

$$a) \lim_{x \rightarrow +\infty} [\sqrt{4x^2 - 3x + 1} - ax + b] = 1/4 \Leftrightarrow (a, b) = (2, 1)$$

$$b) \lim_{x \rightarrow -\infty} [\sqrt[3]{x^3 + 1} - ax - b] = 0 \Leftrightarrow (a, b) = (1, 0)$$

Limit composition theorem

Def : Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$. We define the composition $f \circ g: C \rightarrow \mathbb{R}$ such that

$$\begin{cases} C = \{x \in B \mid g(x) \in A\} \\ \forall x \in C: (f \circ g)(x) = f(g(x)) \end{cases}$$

Note that the belonging condition for the domain C of $f \circ g$ is given by

$$x \in C \Leftrightarrow \begin{cases} x \in B \\ g(x) \in A \end{cases}$$

The following theorems make it possible to calculate the limit of the composition $f \circ g$:

Thm : Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ and $f \circ g: C \rightarrow \mathbb{R}$. Let a be a limit point of C . Then, we have:

$$\begin{cases} \lim_{x \rightarrow a} g(x) = a \in \mathbb{R} \\ \lim_{x \rightarrow a} f(x) = f(a) \end{cases} \Rightarrow \lim_{x \rightarrow a} f(g(x)) = f(a)$$

Proof

Let $\varepsilon \in (0, +\infty)$ be given. Since

$$\lim_{x \rightarrow a} f(x) = f(a) \Rightarrow \exists \delta \in (0, +\infty): \forall x \in A: (0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$$

We choose $\delta_1 \in (0, +\infty)$ such that

$$\forall x \in A : (0 < |x - a| < \delta_1 \Rightarrow |f(x) - f(a)| < \varepsilon)$$

Since:

$$\lim_{x \rightarrow \sigma} g(x) = a \Rightarrow \exists \delta \in (0, +\infty) : \forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \delta_1)$$

choose $\delta \in (0, +\infty)$ such that

$$\forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \delta_1)$$

Let $x \in G$ be given and assume that $x \in N(\sigma, \delta)$. We will show that $|f(g(x)) - f(a)| < \varepsilon$. We have:

$$\begin{aligned} \begin{cases} x \in G \\ x \in N(\sigma, \delta) \end{cases} &\Rightarrow \begin{cases} x \in B \\ x \in N(\sigma, \delta) \end{cases} && [\text{via } G \subseteq B] \\ &\Rightarrow |g(x) - a| < \delta_1 \end{aligned}$$

We need the stronger condition $0 < |g(x) - a| < \delta_1$, so we distinguish between the following cases.

Case 1: Assume that $g(x) = a$. Then, we have

$$|f(g(x)) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon \Rightarrow \underline{|f(g(x)) - f(a)| < \varepsilon}.$$

Case 2: Assume that $g(x) \neq a$. Then, we have:

$$\begin{cases} 0 < |g(x) - a| < \delta_1 \\ g(x) \in A \end{cases} \Rightarrow \underline{|f(g(x)) - f(a)| < \varepsilon}$$

We have thus shown that

$$\begin{aligned} \forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in G : (x \in N(\sigma, \delta) \Rightarrow |f(g(x)) - f(a)| < \varepsilon) \\ \Rightarrow \lim_{x \rightarrow \sigma} f(g(x)) = f(a). \quad \square \end{aligned}$$

If we replace the condition $\lim_{x \rightarrow a} f(x) = f(a)$ (compare with definition of continuity, given $x \rightarrow a$ later) with the more general statement $\lim_{x \rightarrow a} f(x) = L \in \mathbb{R} \cup \{\pm\infty, -\infty\}$, then we need an additional $x \rightarrow a$ assumption to ensure the composition theorem still works:

Prop: Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ and $f \circ g: C \rightarrow \mathbb{R}$, and let σ be a limit point of C. Then, we have: $\left\{ \begin{array}{l} \lim_{x \rightarrow \sigma} g(x) = a \in \mathbb{R} \\ \lim_{x \rightarrow a} f(x) = L \in \mathbb{R} \cup \{\pm\infty, -\infty\} \end{array} \right. \Rightarrow \lim_{x \rightarrow \sigma} f(g(x)) = L$ $\exists \delta \in (0, +\infty) : \forall x \in B \cap N(\sigma, \delta) : g(x) \neq a$

Proof

Let $\varepsilon \in (0, +\infty)$ be given. By hypothesis, choose $\delta_0 \in (0, +\infty)$ such that

$$\forall x \in B \cap N(\sigma, \delta_0) : g(x) \neq a.$$

Since:

$$\lim_{x \rightarrow a} f(x) = L \Rightarrow \exists \delta \in (0, +\infty) : \forall x \in A : (0 < |x - a| < \delta \Rightarrow f(x) \in I(L, \varepsilon))$$

choose $\delta_1 \in (0, +\infty)$ such that

$$\forall x \in A : (0 < |x - a| < \delta_1 \Rightarrow f(x) \in I(L, \varepsilon))$$

Since:

$$\lim_{x \rightarrow \sigma} g(x) = a \Rightarrow \exists \delta \in (0, +\infty) : \forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \delta_1)$$

choose $\delta_2 \in (0, +\infty)$ such that

$$\forall x \in B : (x \in N(\sigma, \delta_2) \Rightarrow |g(x) - a| < \delta_1)$$

Choose $\delta \in (0, +\infty)$ such that $\delta = \min\{\delta_0, \delta_2\}$. Let $x \in C$ be given and assume that $x \in N(\sigma, \delta)$. Then we have

$$x \in N(\sigma, \delta) \cap C \Rightarrow \begin{cases} x \in N(\sigma, \delta_0) \cap B \\ x \in N(\sigma, \delta_2) \cap C \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} g(x) \neq a \\ |g(x) - a| < \delta_1 \\ x \in C \end{cases} \Rightarrow \begin{cases} 0 < |g(x) - a| < \delta_1 \\ g(x) \in A \end{cases} \Rightarrow$$

$$\Rightarrow \underline{f(g(x)) \in I(L, \varepsilon)}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in C : (x \in N(\sigma, \delta) \Rightarrow f(g(x)) \in I(L, \varepsilon))$$

$$\Rightarrow \lim_{x \rightarrow \sigma} f(g(x)) = L. \quad \square$$

For the case $\lim_{x \rightarrow \sigma} g(x) \in \{\pm\infty, -\infty\}$, the previous proof can be modified to show the following statement:

Prop: Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ and $f \circ g: C \rightarrow \mathbb{R}$, and let σ be a limit point of C . Then, we have:

$$a) \begin{cases} \lim_{x \rightarrow \sigma} g(x) = +\infty \\ \lim_{x \rightarrow +\infty} f(x) = L \in \mathbb{R} \cup \{\pm\infty, -\infty\} \end{cases} \Rightarrow \lim_{x \rightarrow \sigma} f(g(x)) = L$$

$$b) \begin{cases} \lim_{x \rightarrow \sigma} g(x) = -\infty \\ \lim_{x \rightarrow -\infty} f(x) = L \in \mathbb{R} \cup \{\pm\infty, -\infty\} \end{cases} \Rightarrow \lim_{x \rightarrow \sigma} f(g(x)) = L$$

Proof : Homework.

→ The following statements are immediate consequences of the limit composition theorems:

1) $\lim_{x \rightarrow +\infty} f(x) = L \Rightarrow \lim_{n \in \mathbb{N}^*} f(n) = L$
2) $\lim_{x \rightarrow x_0} f(x) = L \Leftrightarrow \lim_{h \rightarrow 0} f(x_0 + h) = L$
3) $\lim_{x \rightarrow x_0} f(x) = L \Leftrightarrow \lim_{h \rightarrow 1} f(x_0 h) = L$

→ When the function g is a sequence with $g: \mathbb{N}^* \rightarrow \mathbb{R}$ and we choose $\sigma = +\infty$, then the composition theorems give the following statements:

1) $\left\{ \begin{array}{l} \lim_{n \in \mathbb{N}^*} a_n = x_0 \in \mathbb{R} \\ \lim_{x \rightarrow x_0} f(x) = f(x_0) \end{array} \right. \Rightarrow \lim_{n \in \mathbb{N}^*} f(a_n) = f(x_0)$
2) $\left\{ \begin{array}{l} \lim_{n \in \mathbb{N}^*} a_n = x_0 \in \mathbb{R} \\ \lim_{x \rightarrow x_0} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \\ \forall n \in \mathbb{N}^* : a_n \neq x_0 \end{array} \right. \Rightarrow \lim_{n \in \mathbb{N}^*} f(a_n) = L$

$$3) \begin{cases} \lim_{n \in \mathbb{N}^*} a_n = +\infty \\ \lim_{x \rightarrow +\infty} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \end{cases} \Rightarrow \lim_{n \in \mathbb{N}^*} f(a_n) = L$$

$$4) \begin{cases} \lim_{n \in \mathbb{N}^*} a_n = -\infty \\ \lim_{x \rightarrow -\infty} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \end{cases} \Rightarrow \lim_{n \in \mathbb{N}^*} f(a_n) = L$$

THEORY QUESTIONS

(18) Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$. State the domain and definition of the function composition $f \circ g$.

(19) Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ and $f \circ g: C \rightarrow \mathbb{R}$ and let σ be a limit point of C . Prove that

$$a) \lim_{x \rightarrow \sigma} g(x) = a \in \mathbb{R} \wedge \lim_{x \rightarrow a} f(x) = f(a) \Rightarrow \lim_{x \rightarrow \sigma} f(g(x)) = f(a)$$

$$b) \begin{cases} \lim_{x \rightarrow \sigma} g(x) = a \in \mathbb{R} \\ \lim_{x \rightarrow a} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \\ \exists \delta \in (0, +\infty) : \forall x \in B \cap N(\sigma, \delta) : g(x) \neq a \end{cases} \Rightarrow \lim_{x \rightarrow \sigma} f(g(x)) = L$$

EXERCISES

(20) Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ and $f \circ g: C \rightarrow \mathbb{R}$, let σ be a limit point of C and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Prove that:

$$a) \lim_{x \rightarrow \sigma} g(x) = +\infty \wedge \lim_{x \rightarrow +\infty} f(x) = L \Rightarrow \lim_{x \rightarrow \sigma} f(g(x)) = L$$

$$b) \lim_{x \rightarrow \sigma} g(x) = -\infty \wedge \lim_{x \rightarrow -\infty} f(x) = L \Rightarrow \lim_{x \rightarrow \sigma} f(g(x)) = L$$

② Let $f: A \rightarrow \mathbb{R}$ with $A = (0, +\infty)$. Use the composition theorem to show that

$$a) \lim_{x \rightarrow +\infty} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^+} f\left(\frac{1}{x-a}\right) = L$$

$$b) \lim_{x \rightarrow -\infty} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^-} f\left(\frac{1}{x-a}\right) = L$$

→ Note that you have to show both the " $\Rightarrow$ " and " $\Leftarrow$ " statements using separate arguments.

▼ Trigonometric limits

Limits of trigonometric functions are established via the inequality:

$$\forall x \in (-\pi/2, 0) \cup (0, \pi/2): |\sin x| < |x| < |\tan x|$$

via the limit definition, as follows:

● → Basic trigonometric limits

$$(1) \quad \lim_{x \rightarrow x_0} \sin x = \sin x_0, \quad \forall x_0 \in \mathbb{R}$$

Proof

Let $x_0 \in \mathbb{R}$ be given. Let $\varepsilon \in (0, +\infty)$ be given. Choose $\delta = \min \{ \varepsilon, \pi/2 \}$. Let $x \in \mathbb{R}$ be given and assume that $x \in N(x_0, \delta)$. Then, we have:

$$\begin{aligned} |\sin x - \sin x_0| &= \left| 2 \sin\left(\frac{x-x_0}{2}\right) \cos\left(\frac{x+x_0}{2}\right) \right| = \\ &= 2 \left| \sin\left(\frac{x-x_0}{2}\right) \right| \cdot \left| \cos\left(\frac{x+x_0}{2}\right) \right| \\ &\leq 2 \left| \sin\left(\frac{x-x_0}{2}\right) \right| \leq 2 \left| \frac{x-x_0}{2} \right| = \\ &= 2 \cdot \frac{|x-x_0|}{2} = |x-x_0| \Rightarrow |\sin x - \sin x_0| \leq |x-x_0|. \end{aligned}$$

and it follows that

$$x \in N(x_0, \delta) \Rightarrow 0 < |x - x_0| < \delta \leq \varepsilon \Rightarrow |x - x_0| < \varepsilon \Rightarrow \\ \Rightarrow \underline{| \sin x - \sin x_0 | < \varepsilon}$$

We have thus shown that:

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in \mathbb{R} : (x \in N(x_0, \delta) \Rightarrow | \sin x - \sin x_0 | < \varepsilon)$$

$$\Rightarrow \lim_{x \rightarrow x_0} \sin x = \sin x_0$$

Q

$$\textcircled{2} \quad \boxed{\forall x_0 \in \mathbb{R} : \lim_{x \rightarrow x_0} \cos x = \cos x_0}$$

Proof

Let $x_0 \in \mathbb{R}$ be given. Let $\varepsilon \in (0, +\infty)$ be given. Choose $\delta = \min \{ \varepsilon, \pi/2 \}$. Let $x \in \mathbb{R}$ be given and assume that $x \in N(x_0, \delta)$. Then, we have:

$$\begin{aligned} | \cos x - \cos x_0 | &= \left| 2 \sin \left(\frac{x+x_0}{2} \right) \sin \left(\frac{x_0-x}{2} \right) \right| = \\ &= 2 \left| \sin \left(\frac{x+x_0}{2} \right) \right| \cdot \left| \sin \left(\frac{x_0-x}{2} \right) \right| \\ &\leq 2 \left| \sin \left(\frac{x_0-x}{2} \right) \right| \leq 2 \left| \frac{x_0-x}{2} \right| = \\ &= 2 \frac{|x-x_0|}{2} = |x-x_0| \end{aligned}$$

and it follows that

$$x \in N(x_0, \delta) \Rightarrow 0 < |x - x_0| < \delta \leq \varepsilon \Rightarrow |x - x_0| < \varepsilon \Rightarrow \\ \Rightarrow \underline{| \cos x - \cos x_0 | < \varepsilon}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in \mathbb{R} : (x \in N(\sigma, \delta) \Rightarrow |\cos x - \cos x_0| < \varepsilon)$$

$$\Rightarrow \lim_{x \rightarrow x_0} \cos x = \cos x_0$$

□

Using the limit properties, from the previous two results we immediately obtain:

③	$\forall x_0 \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\} : \lim_{x \rightarrow x_0} \tan x = \tan x_0$
④	$\forall x_0 \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\} : \lim_{x \rightarrow x_0} \cot x = \cot x_0$

→ Upgrades via limit composition theorem

Using the composition theorem, these results can be upgraded to obtain:

①	$\lim_{x \rightarrow \sigma} g(x) = a \in \mathbb{R} \Rightarrow \lim_{x \rightarrow \sigma} \sin(g(x)) = \sin a$
②	$\lim_{x \rightarrow \sigma} g(x) = a \in \mathbb{R} \Rightarrow \lim_{x \rightarrow \sigma} \cos(g(x)) = \cos a$
③	$\lim_{x \rightarrow \sigma} g(x) = a \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\} \Rightarrow \lim_{x \rightarrow \sigma} \tan(g(x)) = \tan a$
④	$\lim_{x \rightarrow \sigma} g(x) = a \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\} \Rightarrow \lim_{x \rightarrow \sigma} \cot(g(x)) = \cot a$

• $\rightarrow$ 0/0 trigonometric limits

The squeeze theorem for function limits follows from the squeeze theorem for convergent nets, and it reads:

Thm: Let $f: A \rightarrow \mathbb{R}$ and $g_1: A \rightarrow \mathbb{R}$ and $g_2: A \rightarrow \mathbb{R}$ and let σ be a limit point of A . Then, we have:

$$\begin{cases} \forall x \in A \cap N(\sigma, \delta): g_1(x) \leq f(x) \leq g_2(x) \Rightarrow \lim_{x \rightarrow \sigma} f(x) = l. \\ \lim_{x \rightarrow \sigma} g_1(x) = \lim_{x \rightarrow \sigma} g_2(x) = l \in \mathbb{R} \end{cases}$$

We now use the squeeze theorem to calculate the following limits:

$$(1) \quad \boxed{\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1}$$

Proof

Define $\forall x \in \mathbb{R}^* : f(x) = (\sin x)/x$. Let $x \in (-\pi/2, 0) \cup (0, \pi/2)$ be given. Then, we have:

$$\sin x, x \text{ equisigned} \Rightarrow x \sin x > 0 \Rightarrow \frac{\sin x}{x} > 0$$

and therefore:

$$f(x) = \frac{\sin x}{x} = \left| \frac{\sin x}{x} \right| = \frac{|\sin x|}{|x|} \leq \frac{|x|}{|x|} = 1$$

and

$$f(x) = \frac{\sin x}{x} = \left| \frac{\sin x}{x} \right| = \frac{|\sin x|}{|x|} \geq \frac{|\sin x|}{|\tan x|} = \left| \frac{\sin x}{\tan x} \right|$$

$$= \left| \frac{\sin x}{(\sin x)/(\cos x)} \right| = \left| \frac{L}{1/(\cos x)} \right| = |\cos x|$$

We have thus shown that

$$\forall x \in (-\pi/2, 0) \cup (0, \pi/2) : |\cos x| \leq f(x) \leq 1 \quad (1)$$

It follows that

$$\lim_{x \rightarrow 0} \cos x = \cos 0 = 1 \Rightarrow \lim_{x \rightarrow 0} |\cos x| = |1| = 1 \Rightarrow$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad [\text{via Eq. (1)}] \quad \square$$

An immediate corollary is:

$$\textcircled{2} \quad \boxed{\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1}$$

Using the composition theorem, these results can be upgraded to give:

$$\boxed{\begin{cases} \forall x \in N(\sigma, \delta) \cap \text{dom}(g) : g(x) \neq 0 \Rightarrow \lim_{x \rightarrow \sigma} \frac{\sin(g(x))}{g(x)} = \lim_{x \rightarrow \sigma} \frac{\tan(g(x))}{g(x)} = 1 \\ \lim_{x \rightarrow \sigma} g(x) = 0 \end{cases}}$$

An immediate consequence of these generalizations is that:

$$\boxed{\begin{aligned} \forall a \in \mathbb{R}^+ : \lim_{x \rightarrow 0} \frac{\sin(ax)}{ax} &= \lim_{x \rightarrow 0} \frac{\tan(ax)}{ax} = 1 \\ \forall a \in \mathbb{R} : \lim_{x \rightarrow a} \frac{\sin(x-a)}{x-a} &= \lim_{x \rightarrow a} \frac{\tan(x-a)}{x-a} = 1 \end{aligned}}$$

THEORY QUESTIONS

(22) Prove the following statements

a) $\forall x_0 \in \mathbb{R}: \lim_{x \rightarrow x_0} \sin x = \sin x_0$

b) $\forall x_0 \in \mathbb{R}: \lim_{x \rightarrow x_0} \cos x = \cos x_0$

c) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

EXERCISES

(23) Use the limit properties to evaluate the following limits (WITHOUT use of the De L'Hospital theorem).

a) $\lim_{x \rightarrow +\infty} \frac{\sin x}{x} = 0$

b) $\lim_{x \rightarrow 0} \frac{\cos x - \cos(5x)}{x \sin x} = 12$

c) $\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{\sin(5x)} = \frac{1}{20}$

d) $\lim_{x \rightarrow 0^+} \frac{2x - \sin x}{\sqrt{1 - \cos x}} = \sqrt{2}$

e) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sin x} = \frac{1}{2}$

f) $\lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{\sqrt{1 + \cos(2x)}} = 0$

g) $\lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + \cos(2x)}}{\sin^2 x} = \frac{\sqrt{2}}{2}$

h) $\lim_{x \rightarrow 0} \frac{1 - \cos^3 x}{x \sin(2x)} = \frac{3}{4}$

i) $\lim_{x \rightarrow 0} \left[\frac{2}{\sin^2 x} - \frac{1}{1 - \cos x} \right] = \frac{1}{2}$

j) $\lim_{x \rightarrow 0} \frac{\sqrt{\cos x} - 1}{x^2} = \frac{-1}{4}$

k) $\forall n \in \mathbb{N}^k: \lim_{x \rightarrow 0} \frac{1 - \cos^n x}{x^2} = \frac{n}{2}$

→ Trigonometric limits with $x \rightarrow \pm\infty$

These limits usually do not exist and we can show that using proof by contradiction as follows:

- ₁ To show a contradiction, assume that $\lim_{x \rightarrow +\infty} f(x) = L$.
- ₂ Define sequences $(a_n), (b_n)$ such that
$$\lim_{n \in \mathbb{N}^+} f(a_n) = L_1 \wedge \lim_{n \in \mathbb{N}^+} f(b_n) = L_2 \wedge L_1 \neq L_2$$
- ₃ Use the composition theorem to show that $L_1 = L_2 = L$ and thus derive a contradiction.

EXAMPLE

Show that $\lim_{x \rightarrow +\infty} \sin x$ does not exist.

Solution

To show a contradiction, assume that $\lim_{x \rightarrow +\infty} \sin x = L$ with $L \in \mathbb{R} \cup \{\pm\infty, -\infty\}$. Define $(a_n), (b_n)$ such that

$$\forall n \in \mathbb{N}^+ : (a_n = 2n\pi \wedge b_n = 2n\pi + \pi/4)$$

Since

$$\lim_{n \in \mathbb{N}^+} a_n = +\infty \wedge \lim_{n \in \mathbb{N}^+} b_n = +\infty \Rightarrow \lim_{n \in \mathbb{N}^+} f(a_n) = \lim_{n \in \mathbb{N}^+} f(b_n) = L$$

We also have:

$$\begin{aligned} L_1 &= \lim_{n \in \mathbb{N}^+} f(a_n) = \lim_{n \in \mathbb{N}^+} f(2n\pi) = \lim_{n \in \mathbb{N}^+} \sin(2n\pi) = \\ &= \lim_{n \in \mathbb{N}^+} \sin 0 = \lim_{n \in \mathbb{N}^+} 0 = 0 \end{aligned}$$

$$l_2 = \lim_{n \in \mathbb{N}^*} f(b_n) = \lim_{n \in \mathbb{N}^*} \sin(2n\pi + \pi/4) = \lim_{n \in \mathbb{N}^*} \sin(\pi/4)$$

$$= \sin(\pi/4) = \sqrt{2}/2$$

It follows that $\lim_{n \in \mathbb{N}^*} a_n \neq \lim_{n \in \mathbb{N}^*} b_n$, which is a contradiction

We conclude that $\lim_{x \rightarrow +\infty} \sin x$ does not exist. $\square$

EXERCISES

(24) Show that the following limits do not exist

a) $\lim_{x \rightarrow 0} \sin(1/x)$

b) $\lim_{x \rightarrow 1} \cos\left(\frac{1}{x-1}\right)$

c) $\lim_{x \rightarrow \pi/2} \frac{x - \pi/2}{\sqrt{1 - \sin x}}$

d) $\lim_{x \rightarrow 0} \left[\frac{1}{x} \sin\left(\frac{1}{x}\right) \right]$

e) $\lim_{x \rightarrow +\infty} [2 \cos(3x) - 1]$

f) $\lim_{x \rightarrow +\infty} \sqrt{3 + \cos(x/2)}$

g) $\lim_{x \rightarrow +\infty} \tan x$

h) $\lim_{x \rightarrow \pi} [2x \tan(x/2) + 3]$